C 6

Physics Department
Univ. of Chicago

ROYAL

COMPOSITIONS

NAME E. FERMI Univ. of Chicago

Quantum mechanics

VERNON ROYAL LINE

OFFICIAL PROGRAM

DATE

FAMILY NAME GIVEN NAME

SECTION________________ ROOM________________

	MON.	RM.	TUES.	RM.	WED.	RM.	THURS.	RM.	FRI.	RM.
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										
11										
12										

The covers of this book have been treated with a special varnish that protects against moisture and rubbing.

NOTICE HOW SMOOTH THEY ARE.

The paper has been specially selected for a smooth writing surface, and uniform quality, which may be identified with the watermark.

Royal Writing

LOOK FOR THE WATERMARK IN THE PAPER.

VERNON ROYAL LINE

MADE IN U. S. A.

Mechanics	Optics
Mass point	Wave packett
Trajectory	Ray
Velocity V	Group velocity v
Variable potential	Variable index of refraction
Variation of energy	Variations of frequency

Trajectory) (1) $$\delta \int \sqrt{W-U}\, ds = 0$$ Maupertuis

Ray) (2) $$\delta \int \frac{ds}{v} = 0$$ Fermat

Velocity of the mass point

(3) $$V = \sqrt{\frac{2}{m}(W-U)}$$

Velocity of the wave group

(4) $$\frac{1}{V} = \frac{d}{d\nu}\frac{\nu}{v}$$

From (1) (2)

(5) $$\frac{1}{v} = f(\nu)\sqrt{W(\nu) - U(x,y,z)}$$

From (3)(4)(5)

$$\frac{1}{\sqrt{\frac{2}{m}(W-U)}} = f\sqrt{W-U} + \nu\left\{ f'\sqrt{W-U} + \frac{f W'}{2\sqrt{W-U}}\right.$$

Hence

$$\begin{cases} f + \nu f' = 0 \\ \sqrt{2m} = \nu f W' \end{cases} \qquad \begin{aligned} f\nu &= c_1 \\ W' &= \frac{\sqrt{2m}}{c_1} = h \\ W &= h\nu + \cancel{\text{cst}} \end{aligned}$$

$$f\nu = \frac{\sqrt{2m}}{h}$$

$$W = h\nu \qquad f = \frac{\sqrt{2m}}{h\nu}$$

---

(6) $$W = h\nu$$

(7) $$v = \frac{h\nu}{\sqrt{2m}} \frac{1}{\sqrt{h\nu - U}}$$

(8) $$V = \sqrt{\frac{2}{m}(h\nu - U)}$$

Wave equation with dispersion law

$$v = v(\nu, x, y, z)$$

$$\psi = \sum \psi_\nu = \sum u_\nu e^{-2\pi i \nu t}$$

$$\Delta \psi_\nu - \frac{1}{v_\nu^2} \frac{\partial^2 \psi_\nu}{\partial t^2} = 0$$

$$\Delta \psi_\nu + 4\pi^2 \frac{\nu^2}{v_\nu^2} \psi_\nu = 0$$

Notation is not clear; Better $\psi = \sum_\nu \psi_\nu = \sum_\nu u_\nu (xyz) e^{-2\pi i \nu t}$

For our special case (7)

$$\Delta \psi_\nu + \frac{8\pi^2 m}{h^2} (h\nu - U) \psi_n = 0$$

but since

$$\frac{\partial \psi_\nu}{\partial t} = - 2\pi i \nu \psi_\nu$$

$$\Delta \psi_\nu - \frac{8\pi^2 m}{h^2} U \psi_n + \frac{4\pi m i}{h} \frac{\partial \psi_n}{\partial t} = 0$$

or

(9) $$\Delta \psi - \frac{8\pi^2 m}{h^2} U \psi + \frac{4\pi m i}{h} \frac{\partial \psi}{\partial t} = 0$$

a solution of frequency ν (Energy $W = h\nu$) is

(10) $$\psi = u e^{-2\pi i \frac{W}{h} t}$$

With

(11) $$\Delta u + \frac{8\pi^2 m}{h^2} (W - U) u = 0$$

From (7) for $U = 0$ $h\nu = \frac{1}{2} m V^2$ we obtain

(12) $$\lambda = \frac{v}{\nu} = \frac{h}{mV}$$

Davisson & Germer give h = Plancks constant

4

From (9)

$$\Delta\psi - \frac{8\pi^2 m}{h^2} U\psi + \frac{4\pi m i}{h} \frac{\partial\psi}{\partial t} = 0 \qquad \Big|\ \psi^*$$

and

$$\Delta\psi^* - \frac{8\pi^2 m}{h^2} U\psi^* - \frac{4\pi m i}{h} \frac{\partial\psi^*}{\partial t} = 0 \qquad \Big|\ \psi$$

$$-\psi\Delta\psi^* + \psi^*\Delta\psi + \frac{4\pi m i}{h}\frac{\partial}{\partial t}(\psi^*\psi) = 0$$

or

(13) $$0 = \frac{\partial}{\partial t}(\psi^*\psi) + \operatorname{div} \frac{h}{4\pi m i}\ (\hbar/2mi)\ (\psi^* \operatorname{grad}\psi - \psi \operatorname{grad}\psi^*)$$

Hence

(14) $$\psi^*\psi = \text{density of probability}$$

(15) $$\frac{h}{4\pi m i}(\psi^* \operatorname{grad}\psi - \psi \operatorname{grad}\psi^*) = \text{density of flow}$$

From (14) the normalisation

(16) $$\int \psi^*\psi \, d\tau = 1$$

Hence in a singular point ψ must ~~never~~ be less ~~more~~ infinite than $\frac{1}{r^{3/2}}$ and at infinite distance it must vanish faster than ~~at least as~~ $1/r^{3/2}$

Nuclear distance in HCl molecule $= 1.27 \times 10^{-8}$

Generalizations:

Mass point on a line (x = abscissa)

(17) $$\frac{\partial^2\psi}{\partial x^2} - \frac{8\pi^2 m}{h^2} U(x)\,\psi + \frac{4\pi m i}{h}\frac{\partial\psi}{\partial t} = 0$$

(18) $$\frac{d^2 u}{dx^2} + \frac{8\pi^2 m}{h^2}(W - U)\,u = 0$$

Rotator with fixed axe (A = mom. of inerzia)

(19) $$\frac{\partial^2\psi(\varphi,t)}{\partial\varphi^2} - \frac{8\pi^2 A}{h^2} U(\varphi)\,\psi + \frac{4\pi A i}{h}\frac{\partial\psi}{\partial t} = 0$$

(20) $$\frac{d^2 u(\varphi)}{d\varphi^2} + \frac{8\pi^2 A}{h^2}(W - U)\,u = 0$$

Point on a spherical surface (spherical coordinates ϑ, φ operator

(21) $$\Lambda = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}$$

(22) $$\frac{1}{r^2}\Lambda\psi - \frac{8\pi^2 m}{h^2} U(\vartheta,\varphi)\,\psi + \frac{4\pi m i}{h}\frac{\partial\psi}{\partial t} = 0$$

(23) $$\frac{1}{r^2}\Lambda u + \frac{8\pi^2 m}{h^2}(W - U)\,u = 0$$

This later

Dumbbell ~~rotating~~ with fixed center of gravity

(24) $$\Lambda\psi - \frac{8\pi^2 A}{h^2} U\psi + \frac{4\pi A i}{h}\frac{\partial\psi}{\partial t} = 0$$

(25) $$\Lambda u + \frac{8\pi^2 A}{h^2}(W - U)\,u = 0$$

6

Better present here general points

$$\sum_i \frac{1}{m_i} \Delta_i \psi \sim \Delta\psi$$

General mechanical system with

(26) $$T = \frac{1}{2}\sum m_{ik}(q)\, \dot{q}_i \dot{q}_k$$

Define

(27) $$ds^2 = \sum m_{ik}(q) dq_i\, dq_k$$

Let Δ_q is Laplace's operator in variety (27)

~~Then~~ (29) $$\Delta_q \psi(q_1 \cdots q_f\, t) - \frac{8\pi^2}{h^2} U \psi + \frac{4\pi i}{h} \dot{\psi} = 0$$

(30) $$\Delta_q u(q_1 \cdots q_f) + \frac{8\pi^2}{h^2}(W - U) u = 0$$

Examples: one dimensional

a) Point on a closed line (length $= a$) with $U = 0$

(31) $$u_l(x) = \frac{1}{\sqrt{a}} e^{\frac{2\pi i l}{a} x}$$

$l =$ any integer

(32) $$W_l = \frac{h^2 l^2}{2a^2 m}$$

b) Rotator with fixed axe with $U = 0$

(33) $$u_l(\varphi) = \frac{1}{\sqrt{2\pi}} e^{i l \varphi}$$

(34) $$W_l = \frac{h^2 l^2}{8\pi^2 A}$$

c) Limiting condition where the potential becomes infinite

$$\frac{u'}{u} = -\sqrt{\frac{8\pi^2 m}{h^2} U_\infty}$$

i.e.

(35) $$u = 0$$

d) Point moving on a segment (length $\underline{a}$) with $V=0$ from $x=0$ to $x=a$

(36) $$u_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} x$$

(37) $$W_n = \frac{h^2 n^2}{8 a^2 m}$$

e) Point on an infinite line with $V=0$

As a limit from a) or d) we have

Normalization

(38) $$u_\omega(x) \sim e^{i\omega x}$$

$\int_{-\infty}^{\infty} |e^{i\omega x}|^2 dx = \infty$ but

$\int_{-\infty}^{\infty} |u_{\Delta\omega}|^2 dx$ = finite with

$u_{\Delta\omega} = \int_{\omega}^{\omega+\Delta\omega} e^{i\omega x} dx$

(39) $$W = \frac{h^2 \omega^2}{8\pi^2 m}$$

Hence

(40) $$\lambda = \frac{2\pi}{\omega} = \frac{h}{\sqrt{2mW}} = \frac{h}{mV}$$ (With $W = \frac{1}{2} m V^2$)

f) Harmonic oscillator

(41) $$U = 2\pi^2 \nu^2 m x^2$$

$$U = \frac{\omega^2}{2} m x^2$$

8

$$\frac{d^2u}{dx^2} + \frac{2m}{\hbar^2}\left(W - \frac{\omega^2 m}{2}x^2\right)u$$

(42) $$\frac{d^2u}{dx^2} + \frac{8\pi^2 m}{h^2}\left(W - 2\pi^2 m\nu^2 x^2\right)u = 0$$

With $\xi = \sqrt{\frac{m\omega}{\hbar}}x$ $\qquad \varepsilon = \frac{2W}{\hbar\omega}$

(43) $$\xi = 2\pi\sqrt{\frac{m\nu}{h}}\,x \; ; \quad \varepsilon = \frac{2W}{h\nu}$$

(44) $$\frac{d^2u}{d\xi^2} + \left(\varepsilon - \xi^2\right)u = 0$$

(45) $$u = v(\xi)\, e^{-\xi^2/2}$$

(46) $$\frac{d^2v}{d\xi^2} - 2\xi\frac{dv}{d\xi} + (\varepsilon - 1)\,v = 0$$

Put

(47) $$v = \sum a_r \xi^r$$

recursion formula

(48) $$a_{r+2} = \frac{2r+1-\varepsilon}{(r+1)(r+2)}\,a_r$$

behaves essentially like e^{ξ^2} if not $\varepsilon = 2n+1$

Eigenvalues

$$\varepsilon = 2n+1$$

(49) $$W = h\nu n + \frac{h\nu}{2}$$ $\qquad W = \hbar\omega\left(n + \frac{1}{2}\right)$

First polinomials of Hermite

(50) $$\begin{cases} H_0 = 1 & H_1 = 2\xi \\ H_2 = -2 + 4\xi^2 & H_3 = -12\xi + 8\xi^3 \end{cases}$$

Eigenfunctions:

$$(51) \qquad u_n(\xi) = H_n(\xi)\, e^{-\xi^2/2}$$

General definition of the Hermite polynomials

$$(52) \qquad H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{dx^n} e^{-\xi^2}$$

Proof that:

$$(54) \qquad H''_n - 2\xi H'_n + 2n H_n = 0$$

(54) is equivalent to

$$(55) \qquad \frac{d^{n+2}}{d\xi^{n+2}} e^{-\xi^2} + 2\xi \frac{d^{n+1}}{d\xi^{n+1}} e^{-\xi^2} + (2+2n) \frac{d^n}{d\xi^n} e^{-\xi^2} = 0$$

This can be verified for $n=0$; successive derivations yield it for $n = 1, 2, \ldots$

Property

$$(56) \qquad \frac{d}{d\xi} H_n = 2n H_{n-1}$$

is equivalent to (55) written for $n-1$ instead of n.

From (52)(56)

$$(57) \qquad \int H_m^2(x)\, e^{-x^2} dx = \sqrt{\pi}\, 2^m\, m!$$

The coeff. of normalisat. of solution (51)(43) is then

$$(58) \qquad \left(\frac{4\pi^2 m \nu}{h}\right)^{1/4} \frac{1}{\sqrt{\sqrt{\pi}\, 2^n\, n!}} = \left(\frac{m\omega}{\hbar}\right)^{1/4} \frac{1}{\sqrt{\sqrt{\pi}\, 2^m\, m!}}$$

Orthog follows also immediately from (51)(56).

For

~~Point in a central field of force $V(r)$~~

$$\cancel{\Delta\psi + \frac{8\pi^2 m}{h^2}\left(W - V(r)\right)u = 0}$$

~~Develop~~

$$\cancel{u(r,\vartheta,\varphi) = \sum_{l\,m} v_r\, Y_{l,m}(\vartheta,\varphi)}$$

~~with~~

$$\cancel{Y_{lm}(\vartheta,\varphi) =}$$

Spherical Harmonics and Legendre polinomials

(59) $$Y_{lm}(\theta,\varphi) = \frac{1}{N_{lm}}\, e^{im\varphi} \sin^{|m|}\vartheta\, \frac{d^{|m|} P_l(\cos\vartheta)}{d(\cos\vartheta)^{|m|}}$$

(60) $$\frac{1}{N_{lm}} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(l-|m|)!}{(l+|m|)!}\,\frac{2l+1}{2}}$$

(61) $$P_l(x) = \frac{1}{2^l\, l!}\,\frac{d^l}{dx^l}\,(x^2-1)^l$$

(62) $$(1-x^2)\,P'' - 2x\,P' + l(l+1)\,P = 0$$

(63) $$\int_{-1}^{1} P_l^2(x)\,dx = \frac{2}{2l+1} \qquad \int_{-1}^{1} P_l(x)\,P_{l'}(x)\,dx = 0 \text{ if } l \neq l'$$

(64) $$P_l = \frac{2l-1}{l}\,x \cdot P_{l-1} - \frac{l-1}{l}\,P_{l-2}$$

(65) $$\begin{cases} P_0 = 1 & P_1 = x \\ P_2 = \frac{3}{2}x^2 - \frac{1}{2} & P_3 = \frac{5}{2}x^3 - \frac{3}{2}x \\ P_4 = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8} & P_5 = \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x \end{cases}$$

Spherical functions satisfy diff. equation

(66) $$\Lambda Y_{lm}(\vartheta,\varphi) = -l(l+1)Y_{lm}(\vartheta,\varphi)$$

and the orthogonality relation

(67) $$\int_{4\pi} Y^*_{lm}\, Y_{l'm'}\, d\omega = \delta_{ll'}\,\delta_{mm'}$$

Point on spherical surface without forces: $U=0$

Schrödinger equation (23). Put $\Lambda u + \frac{2I}{\hbar^2} W u = 0$

(68) $$u = \sum c_{lm} Y_{lm}$$

Substitute in (23) [with (66)]

(69) $$\sum_{lm} c_{lm} Y_{lm}\left[-\frac{l(l+1)}{r^2} + \frac{8\pi^2 m}{h^2} W\right] = 0$$

hence non vanishing solution only for

(70) $$W = \frac{h^2 l(l+1)}{8\pi^2 m r^2} = \frac{\hbar^2 l(l+1)}{2m r^2}$$

A dumbbell with no forces (25) has similarly

(71) $$W = \frac{h^2 l(l+1)}{8\pi^2 A} = \frac{\hbar^2 l(l+1)}{2A}$$

$$\nabla^2 u(r,\vartheta,\varphi) = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Lambda u$$

$$\Lambda u = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2}$$

Central forces

(72) $$U = U(r)$$

(73) $$\Delta u + \frac{8\pi^2 m}{h^2}\left(W - U(r)\right) u = 0$$

$\frac{2m}{\hbar^2}$

(74) $$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \frac{1}{r^2}\Lambda u + \frac{8\pi^2 m}{h^2}\left(W - U(r)\right) u = 0$$

(75) $$u = \sum R_{lm}(r)\, Y_{l,m}(\vartheta, \varphi)$$

(76) $$\sum_{lm} Y_{l,m}(\vartheta,\varphi)\left[R''_{lm} + \frac{2}{r}R'_{lm} - \frac{l(l+1)}{r^2}R_{lm} + \frac{8\pi^2 m}{h^2}\left(W - U(r)\right)R_{lm}\right] =$$

(77) $$R''_l + \frac{2}{r}R'_l + \frac{8\pi^2 m}{h^2} W R_l - \left\{\frac{8\pi^2 m}{h^2}U(r) + \frac{l(l+1)}{r^2}\right\} R_l = 0$$

(78) $$y'' + \left[\frac{8\pi^2 m}{h^2}W - \frac{8\pi^2 m}{h^2}U - \frac{l(l+1)}{r^2}\right] y = 0 \; ; \; y = R_l / r$$

Energy levels are all the e.v. of all the (77)

Coulomb forces

(79) $$U(r) = -\frac{Ze^2}{r}$$

$$\cancel{R''_l + \frac{2}{r}R'_l + \frac{8\pi^2 m W}{h^2}R_l + \left\{\frac{8\pi^2 m}{h^2}\frac{Ze^2}{r} - \frac{l(l+1)}{r^2}\right\} R_l = 0}$$

(80) $$y'' + \left[\frac{8\pi^2 m}{h^2}W + \frac{8\pi^2 m Ze^2}{h^2 r} - \frac{l(l+1)}{r^2}\right] y = 0$$

put

(81) $$r_0 = \sqrt{\frac{h^2}{8\pi^2 m |E|}} \; ; \quad x = 2\frac{r}{r_0}$$

$R_l Y_{l,m}$ $\quad R_l = r\,y(r)$ $\quad x = \frac{2r}{r_0}$ $\quad r_0 = \sqrt{\frac{\hbar^2}{2m|E|}}$

(82) $$y'' + \left[\pm\frac{1}{4} + \frac{A}{x} - \frac{l(l+1)}{x^2}\right] y = 0$$

(83) $$A = \frac{Ze^2}{2 r_0 |E|} = \frac{Ze^2}{2|E|}\sqrt{\frac{2m|E|}{\hbar^2}} = \sqrt{\frac{mZ^2e^4}{2\hbar^2|E|}}$$

First case:

$$W < 0$$

(84) $$y'' + \left[-\frac{1}{4} + \frac{A}{x} - \frac{l(l+1)}{x^2}\right] y = 0$$

(85) $$y = e^{-\frac{x}{2}} v(x)$$

(86) $$v'' - v' + \left(\frac{A}{x} - \frac{l(l+1)}{x^2}\right) y = 0$$

(87) $$v = x^{l+1} w(x)$$

(88) $$w_s = \sum_0^\infty a_s x^s$$

(89) $$a_{s+1} = a_s \frac{s+l+1-A}{(s+1)(s+2l+2)}$$

(90) $$A = n = n' + l + 1$$

(91) $$W_n = -\frac{2\pi^2 m Z^2 e^4}{h^2 n^2} = -\frac{mZ^2e^4}{2\hbar^2 n^2}$$

$$n = l+1,\ l+2,\ l+3, \ldots$$

Laguerre's polinomials

$$L_k(x) = e^x \frac{d^k}{dx^k}\left(x^k e^{-x}\right) \tag{92}$$

$$\left\{\begin{aligned} L_0 &= 1 \\ L_1 &= 1 - x \\ L_2 &= 2 - 4x + x^2 \\ L_3 &= 6 - 18x + 9x^2 - x^3 \end{aligned}\right. \tag{93}$$

Put

$$f(x) = x^k e^{-x} \;; \qquad L_k = e^x f^{(k)}(x) \tag{94}$$

$$x f' = (k - x) f$$

$$x f^{(k+2)} + (1 + x) f^{(k+1)} + (k+1) f^{(k)} = 0$$

$$x L''_k + (1 - x) L'_k + k L_k = 0 \tag{95}$$

From (92) follows besides

$$\int_0^\infty L_k(x)\, L_{k'}(x)\, e^{-x}\, dx = 0 \quad \text{for } k \neq k' \tag{96}$$

Differentiating (95) j times

$$x L_k^{(j)\,''}(x) + (j + 1 - x) L_k^{(j)\,'}(x) + (k - j) L_k^{(j)} = 0 \tag{97}$$

$$L_k^{(j)} = \frac{d^j}{dx^j}\left\{e^x \frac{d^k}{dx^k}\left(x^k e^{-x}\right)\right\} \tag{98}$$

Orthogonality

$$\int_0^\infty L_k^{(j)} L_{k'}^{(j)}\, x^j e^{-x}\, dx = \frac{(k!)^3}{(k-j)!}\, \delta_{kk'}\,. \tag{99}$$

Potential energy $= -\frac{Ze^2}{r}\left(1+\frac{\beta}{r}\right)$

(82) becomes (for the case $E<0$

(82)' $y'' + \left[-\frac{1}{4} + \frac{A}{x} + \frac{2A\beta}{r_0 x^2} - \frac{l(l+1)}{x^2}\right] y = 0$

put

$$l'(l'+1) = l(l+1) - \frac{2A\beta}{r_0}$$

(82)'' $$= l(l+1) - \frac{2\beta}{a}$$

$$a = \frac{\hbar^2}{me^2 Z}$$

Eigenvalues

$$A = \overset{\text{integer}}{n''} + l' + 1 = n' + l + 1 - (l - l')$$

$$= n - (l - l') = n - \alpha_l$$

(82)''' $$E = -\frac{me^4 Z^2}{2\hbar^2 [n - \alpha_l]^2}$$

Wentzel Kramers Brillouin

$$\psi'' + \frac{2m}{\hbar^2}(E-U)\psi = 0$$

$$\psi'' + g\,\psi = 0$$

$$\psi = e^{iy} \qquad \psi' = i y' e^{iy}$$

$$\psi'' = (-y'^2 + i y'')e^{iy}$$

$$y'^2 - i y'' = g$$

$$y' = \sqrt{g} + \varepsilon$$

$$\cancel{g} + 2\sqrt{g}\,\varepsilon - \frac{i g'}{2\sqrt{g}} = 0$$

$$\varepsilon = -\frac{i g'}{4g}$$

$$y = \int \sqrt{g}\,dx - i \log g^{1/4}$$

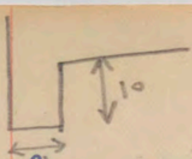
$$\psi \cong \frac{1}{g^{1/4}} e^{i\int\sqrt{g}\,dx}$$

$$\omega(x) = \begin{cases} \frac{1}{x^{1/4}}\cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right) \\ \frac{1}{2(-x)^{1/4}} e^{-\frac{2}{3}(-x)^{3/2}} \end{cases}$$

$$\omega'' + x\omega = 0$$

$$\int_A^B \sqrt{\frac{2m}{\hbar^2}(E-U)}\,dx = \pi\left(n + \frac{1}{2}\right) \qquad n = 0, 1, 2, \ldots$$

(58)



$E_0 = 5.4$

Expression of the e.f. of hydrogenlike systems

$$(100) \quad R_{nl}(r) = \sqrt{\frac{4(n-l-1)!}{a^3 n^4 [(n+l)!]^3}}\, e^{-\frac{r}{na}} \left(\frac{2r}{na}\right)^l L_{n+l}^{(2l+1)}\left(\frac{2r}{na}\right)$$

with

$$(101) \quad a = \frac{h^2}{4\pi^2 m Z e^2} = \frac{\hbar^2}{m e^2}\frac{1}{Z}$$

$$(102) \quad u_{n,l,m}(r,\vartheta,\varphi) = R_{nl}(r)\, Y_{l,m}(\vartheta,\varphi)$$

$$1s = \frac{1}{\sqrt{\pi a^3}} e^{-\sigma}$$

$$2s = \frac{(2-\sigma) e^{-\frac{\sigma}{2}}}{4\sqrt{2\pi a_0^3}}$$

$$2p = \frac{\sigma e^{-\sigma/2} \cos\vartheta}{4\sqrt{2\pi a_0^3}}$$

$$\boxed{\sigma = r/a}$$

Orthogonality of wave functions

$$(103) \quad + \frac{d^2 u_n}{dx^2} + \frac{8\pi^2 m}{h^2}\left(W_n - U(x)\right) u_n = 0$$

$$(104) \quad \frac{d^2 u_m}{dx^2} + \frac{8\pi^2 m}{h^2}\left(W_m - U(x)\right) u_m = 0$$

$$(105) \quad \frac{d}{dx}\left(u_m u_n' - u_n u_m'\right) + \frac{8\pi^2 m}{h^2}\left(W_n - W_m\right) u_n u_m = 0$$

With boundary condition $\cancel{u(a) = u(b) = 0}$

$$(106) \quad u(a) = u(b) = 0$$

follows

$$(107) \quad (W_n - W_m)\int_a^b u_n u_m \, dx = 0$$

$\cancel{u(1s) = \frac{e^{-r/a}}{\sqrt{\pi a^3}}}$; $\cancel{u(2s) = \frac{1}{4\sqrt{2\pi a^3}} e^{-r/2a}\left(2 - \frac{r}{a}\right)}$; $\cancel{u(2p) \propto r e^{-\frac{r}{2a}} Y_{1m}}$

Problems on H atom

analogous proof for three dimensions

(108) $$(W_n - W_m)\int u_n^* u_m \, d\tau = 0$$

Cases of degeneracy (e.g. twofold) by choice of a suitable base

(109) $$\int u_n^* u_m \, d\tau = \delta_{nm}.$$

Coefficients of the expansion

(110) $$u = \sum a_r u_r$$

(111) $$a_r = \int u_r^* u \, d\tau$$

If the expansion is possible

(112) $$\sum a_r u_r e^{-\frac{2\pi i}{h} W_r t}$$

Is the general integral of the time dependent Schroedinger equation (~~Discussion of~~

Determination of the solution of the time dep. Sch. eq. requires knowledge of

(113) $$\psi(x, y, z, 0)$$

Assumption

$|a_n|^2$

is (proportional to) the probability that the energy has value W_n

~~Operators~~ Definitions

~~Linear Operators~~

~~Lin~~ Operator

a Examples $(\ )^2$, $(\ )^3$, 3, $\frac{d}{dx}$, $\frac{d^2}{dx^2}$, $(7x^2+1)$

Field in which the functions are defined

b Examples x, $x\,y\,z$, spherical surface, discreet (finite or infinite) set of points, spaces or varieties of any number of dimensions

c Linear operators

d Operators of ~~fields~~ discreet fields as linear substitution; Matrices

e Linear operators as linear vector transformations in the Hilbert space

f Sum, difference, product of operators

h Eigenvalues + eigenfunctions of a linear

18

operator

$$A u_n = a_n u_n$$

Examples

i (11.4) $$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial \varphi}$$

(115) $$a_n = ni$$

(116) $$x^2 - \frac{d^2}{dx^2}$$

j ~~(117)~~ $$a_n = 2n+1$$

K ~~Herm Operato~~ Secular equation for the e.v.

Hermitian operators for finite matrixes

l (117) $$a_{rs} = a^*_{sr}$$

m Hermitian operators have real eigenvalues and orthogonal e.f. (case of field of n points)

g Sum and product of two operators that can be represented by finite matrixes n Symmetrised product is Hermitian

---

o ~~Property of Hermite operators~~

~~(118)~~ $$\sum v_r^* (A u)_r = \sum u_r (A v)_r^*$$ equivalent to definition

~~or~~

~~(119)~~ $$\int v^* (A u)\, d\tau = \int u\, (A v)^* d\tau$$

~~Definition in the latter case, equivalent to definition for finite matrixes~~

---

p From (119) follows: Reality of e.v. and orthogonality of e.f.
Indeed

(120) $$\int v^* A v\, d\tau = \text{real}$$

---

q Examples:

(121) $i \frac{\partial}{\partial x}$ is hermitian

---

q' Development in a series of e.f.
Geometrical discussion

---

q'' Functions of operators

To each physical magnitude (real)

(122) $$F(x, y, z\ p_x\ p_y\ p_z)$$

there is a corresponding hermithian operator

(123) $$\mathsf{F}$$

obtained by

(124) $$p_x = \frac{h}{2\pi i}\frac{\partial}{\partial x}, \dots$$

M Several physical magnitudes being given it is possible to find for them simultaneously the values

(125) $$\begin{cases} F = a \\ G = b \\ \dots \end{cases}$$

Provided the system

(126) $$\begin{cases} \mathsf{F}\psi = a\psi \\ \mathsf{G}\psi = b\psi \end{cases}$$

see also page 100

has at least one solution

s The (125) define the state of the system when the solution of (125) is unique

t The state of the system at a given time can also be described by giving

ψ

at that time (Measuring of F shall give then $F = a$)

The time variation of ψ is according to the time dependent Schrödinger equation

$$(127) \qquad H\psi = -\frac{h}{2\pi i}\frac{\partial\psi}{\partial t}$$

If ψ is an e.f of F then

$$(128) \qquad F = \text{corresp. e.v.}$$

otherwise develop ψ in e.f of F

$$(129) \qquad \psi = \sum c_n f_n$$

If everything is normalised

$$(130) \qquad |c_n|^2$$

is the probability for the e.v. number $\underline{n}$

Jump to after (147) → (167) then insert what follows

to this shift after discussion of the uncertainty relation formula (167)

Discussion of the system mass point on a straight line

If G' is a not degenerate e.v. of G then

$$(131) \qquad G = G'$$

determines the state as that for which

$$(132) \qquad G\psi = G'\psi$$

Vice versa if the state at $t=0$ is given by

(133) $$\psi(x,0) = \rho(x) e^{i\vartheta(x)}$$

Put

(134) $$G = \left(p - \frac{h}{2\pi}\frac{d\vartheta}{dx}\right)^2 + \frac{h^2}{4\pi^2}\frac{1}{\rho}\frac{d^2\rho}{dx^2}$$

Then

(135) $$G\psi = 0$$

or measure of G gives result

(136) $$G = 0$$

Given $G(x,p)$ how to measure its value for $t=t_1$ by a measurement performed at $t=0$

The equation

(137) $$G\psi(x,t_1) = g\psi(x,t_1)$$

determines

(138) $$\psi(x,t_1,g)$$

integrating back time dependent Schrödinger eq find

(139) $$\psi(x,0,g)$$

With (134) find such an operator G_0 that

(140) $$G_0(x,p,g)\psi(x,0,g) = 0$$

Then

(141) $$G = g \quad \text{at } t = t_1$$

is equivalent to

(142) $$G_0(x, p, g) = 0$$

Solving this with respect to g

$$g = A(x, p)$$

$A(x, p)$ is a magnitude that, measured at $t = 0$ gives as value the same as G measured at $t = t_1$

Let e.g. put for the case of no forces

(143) $$G = x \qquad g = x_1$$

(144) $$\psi(x, t_1) = \delta(x - x_1)$$

(145) $$\frac{\partial^2 \psi}{\partial x^2} = -\frac{4\pi m i}{h}\frac{\partial \psi}{\partial t}$$

corresponding solution

(146) $$\psi(x, t) = \lim_{\alpha = +0} \frac{1}{\sqrt{t_1 - t}} e^{-\frac{(x - x_1)^2}{t_1 - t}\left[\alpha + \frac{\pi i m}{h}\right]}$$

~~so~~ hence

(147) $$\psi(x, 0) \cancel{=} \cancel{\frac{1}{\sqrt{t_1}}} \text{ prop to } e^{-\frac{\pi i m}{h t_1}(x_1 - x)^2}$$

---

follows after (167)

The uncertainty principle
Example: discussion of electronic microscope

$$\cancel{\left[(A-a)^2 + (B-b)^2\right]\psi = 0}$$

$$\cancel{(A-a)^2\psi = -(B-b)^2\psi}$$

Wave mechanical interpretation:

$$\psi(x) = \int A(p)\, e^{\frac{2\pi i}{h} p x}\, dp \tag{148}$$

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \iiint A^*(p)\, A(p')\, e^{\frac{2\pi i}{h}(p'-p)x}\, dp\, dp'\, dx \tag{149}$$

$$\boxed{\int_{-\infty}^{\infty} e^{i\omega x}\, dx = 2\pi\,\delta(\omega)} \tag{150}$$

$$N = \int_{-\infty}^{\infty} |\psi|^2 dx = 2\pi \iint A^*(p)\, A(p')\, \delta\left[\frac{2\pi}{h}(p'-p)\right] dp'\, dp \tag{151}$$

$$= h \int_{-\infty}^{\infty} |A(p)|^2\, dp$$

(152) $$\bar{x} = \frac{1}{N}\int x\,|\psi|^2\,dx$$

(153) $$\bar{p} = \frac{h}{N}\int p\,|A(p)|^2\,dx$$

(154) $$\Delta x^2 = \frac{1}{N}\int (x-\bar{x})^2\,|\psi|^2\,dx$$

(155) $$\Delta p^2 = \frac{h}{N}\int (p-\bar{p})^2\,|A(p)|^2\,dp$$

Take $\bar{x}$ as origin and put

(156) $$\begin{cases} \varphi(x) = \psi(x)\,e^{-\frac{2\pi i}{h}\bar{p}x} \\ B(p) = A(\bar{p}+p) \end{cases}$$

then for the new functions

(157) $$\varphi(x) = \int B(p)\,e^{\frac{2\pi i}{h}px}\,dp$$

&

(158) $$\bar{x} = 0 \qquad \bar{p} = 0$$

(159) $$\Delta x^2 = \frac{1}{N}\int x^2\,|\varphi|^2\,dx$$

(160) $$\Delta p^2 = \frac{h}{N}\int p^2\,|B|^2\,dp$$

(161) $$D = \left|\frac{x}{2\Delta x^2}\varphi + \frac{\partial\varphi}{\partial x}\right|^2 \geq 0$$

26

$$0 \leq \int D\,dx = \int \left(\frac{x^2}{4\Delta x^4} \varphi^* \varphi + \frac{1}{2\Delta x^2} \frac{\partial}{\partial x}(x \varphi^* \varphi) - \frac{1}{2\Delta x^2} \varphi^* \varphi + \right.$$

$$\left. + \frac{\partial}{\partial x}\left(\varphi^* \frac{\partial \varphi}{\partial x}\right) - \varphi^* \frac{\partial^2 \varphi}{\partial x^2} \right) dx$$

(162) $$= -\frac{N}{4\Delta x^2} - \int \varphi^* \frac{\partial^2 \varphi}{\partial x^2}\,dx \geq 0$$

(163) $$\begin{cases} \int \varphi^* \frac{\partial^2 \varphi}{\partial x^2}\,dx = -\frac{4\pi^2}{h^2} \int p^2 B^*(p') B(p) e^{\frac{2\pi i}{h}(p-p')x}\,dx\,dp\,dp' \\ = -\frac{4\pi^2}{h} \iint p^2 B^*(p') B(p)\, \delta(p-p')\,dp\,dp' \\ = -\frac{4\pi^2}{h} \int p^2 \left|B(p)\right|^2 dp \\ = -\frac{4\pi^2 N}{h^2} \Delta p^2 \end{cases}$$

follows

(164) $$-\frac{1}{4\Delta x^2} + \frac{4\pi^2}{h^2} \Delta p^2 \geq 0$$

or

(165) $$\Delta x\, \Delta p \geq \frac{h}{4\pi}$$

back to page 20

Probability distribution of the results of a measurement when state is previously known

A has e.v. $a_1\, a_2 \dots a_n \dots$ & e.f. $\varphi_1\, \varphi_2 \dots \varphi_n \dots$

(166) $$\left| \int \varphi_n^* \psi \, dx \right|^2$$

is proportional to probability of $A = a_n$
If by actual measurement we find $A = a_n$
the state of the system is changed to $\varphi_n(x)$

Time dependent S.e. is description of variation of state. Can be written

(167) $$H\psi = -\frac{h}{2\pi i}\frac{\partial \psi}{\partial t}$$

Insert here what follows (134)

Question:
The state of a mass point is

$$\psi(x) = e^{-i\left(\frac{x^2}{2} - 2x\right)}$$

find

$$(p + x = 2)$$ having put $\frac{h}{2\pi} = 1$

& distributions of p & x

follows after (147)

According to (134) the operator G_0 is

(168) $$p + \frac{2h}{2\pi}\frac{\pi m}{h t_1}(x - x_1)$$

& $G_0 = 0$ is equivalent to

(169) $$x_1 = x + \frac{t_1 p}{m}$$

[Page 66 & commutation rules

$$p_x x - x p_x = \frac{\hbar}{i}$$

Time derivative of an operator

(170) $$H\psi = -\frac{h}{2\pi i}\frac{\partial\psi}{\partial t}$$

(171) $$\psi(x,\tau) = \psi(x,0) - \frac{2\pi i}{h}\tau H\psi(x,0)$$
$$= \left(1 - \frac{2\pi i\tau}{h}H\right)\psi(x,0)$$

notice

(172) $$\boxed{\begin{cases} \psi(x,t) = e^{-\frac{2\pi i t}{h}H}\psi(x,0) \\ \psi(x,0) = e^{\frac{2\pi i t}{h}H}\psi(x,t) \end{cases}}$$

$$A(t) = e^{\frac{it}{\hbar}H} A e^{-\frac{it}{\hbar}H}$$

(173) $$\begin{cases} A\psi(x,t) = \varphi(x,\tau) \\ A(\tau)\,\psi(x,0) = \varphi(x,0) \end{cases}$$

(174) $$A(\tau) = \left(1 + \frac{2\pi i\tau}{h}H\right)A\left(1 - \frac{2\pi i\tau}{h}\right)$$

(175) $$\frac{dA}{dt} = \frac{2\pi i}{h}(HA - AH)$$

relation with Hamilton equations

$$\cancel{\psi(x,\tau) = \sum c_n \psi_n}$$

$$\cancel{A(\tau)\psi(x,0) = \sum c_n}$$

$$\cancel{\psi(x,0) = \sum c_n\left(1 + \frac{2\pi i\tau}{h}H\right)\psi_n}$$

$$\cancel{A(\tau)\left(1 + \frac{2\pi i\tau}{h}H\right)\psi_n = a_n\left(1 + \frac{2\pi i\tau}{h}H\right)\psi_n}$$

Average of a physical magnitude

(177) $$c_n = \int \psi_n^* A \psi$$

(178) $$\psi = \sum c_n \psi_n$$

(179) $$A\psi = \sum c_n a_n \psi_n$$

(180) $$\int \psi^* A \psi = \sum |c_n|^2 a_n$$

$$A\, \psi(x, \tau) = \sum c_n \psi_n$$

$$c_n \,,\ a_n$$

(181) $$\psi(x, 0) = \sum \left(1 + \frac{2\pi i \tau}{h} H\right) \psi_n c_n$$

(182) $$A(\tau) = \left(1 + \frac{2\pi i \tau}{h} H\right) A \left(1 - \frac{2\pi i \tau}{h} H\right)$$

has the same e.v as A

a time constant A obeys to

(183) $$AH = HA$$

as example $M_z = x p_y - y p_x$ is constant in central field of force. ~~+ Hamilton equations~~

Physical meaning of the commutation of A + B

Theorem [see page 66]

If A and B have the same set of e.f. they commute

(184) $\begin{cases} \varphi_1 \dots \varphi_n \dots \\ a_1 \dots a_n \dots \end{cases}$

(185) $\{ b_1 \dots b_n \dots$

(186) $\psi = \Sigma c_n \varphi_n$

(187) $AB\,\psi = \Sigma c_n a_n b_n \varphi_n = BA\,\psi$

If there is no degeneracy the value of A determines the value of B of

(188) $B = f(A)$ Provided $AB = BA$

Integration of time dependent S.E.

(189) $$H\psi = -\frac{h}{2\pi i}\frac{\partial \psi}{\partial t}$$

(190) $$\psi = \Sigma a_n \psi_n$$

(191) $$H\psi_n = w_n \psi_n$$

(192) $$-\frac{h}{2\pi i}\dot{a}_n = w_n a_n$$

(193) $$a_n = a_{n_0} e^{-\frac{2\pi i}{h} w_n t}$$

(194) $$\psi = \Sigma a_{n_0} e^{-\frac{2\pi i}{h} w_n t} \psi_n$$

Transformation theory

Unitarian transformations

$\cancel{T_{\alpha n}}$

(195) $$\sum_n T_{\alpha n} a_n = a_\alpha$$

(196) $$\sum a_n \psi_n = \sum a_\alpha \varphi_\alpha = \psi$$

(197) $$a_n = \int \psi_n^* \psi = \sum a_\alpha \int \psi_n^* \varphi_\alpha$$

(198) $$a_\alpha = \int \varphi_\alpha^* \psi = \sum a_n \int \varphi_\alpha^* \psi_n$$

(199) $$T_{\alpha n} = \int \varphi_\alpha^* \psi_n \qquad T_{\alpha n}^* = T_{n\alpha}$$

(200) $$T_{n\alpha} = \int \psi_n^* \varphi_\alpha$$

$\cancel{\sum \varphi_\alpha^* \varphi_\alpha =}$

(201) $$\delta(x - x_1) = \sum \varphi_\alpha^*(x_1)\, \varphi_\alpha(x)$$

(202) $$\sum_\alpha T_{n\alpha} T_{\alpha m} = \iint \psi_n^*(x_1)\, \varphi_\alpha(x_1)\, \varphi_\alpha^*(x_2)\, \psi_m(x_2)$$
$$= \int \psi_n^*(x_1)\, \psi_m(x_1) = \delta_{nm}$$

$$T\tilde{T} = 1$$

Hamilton equations

From

(203) $$pq - qp = \frac{h}{2\pi i}$$

follows

(204) $$F(p,q)\,q - q\,F(p,q) = \frac{h}{2\pi i}\frac{\partial F}{\partial p}$$

(205) $$p\,F(p\,q) - F(p,q)\,p = \frac{h}{2\pi i}\frac{\partial F}{\partial q}$$

proof: If (204) (205) hold for p & q; if they hold for two functions they hold also for sum & product.

From (175) follow Hamilton equations hence (203) as general commutation rules for general coordinates & momenta

---

Theorem:

If two not degenerate operators A & B commute they have equal sets of e.f.

Let

(206) $$\varphi_1\,\varphi_2 \cdots \varphi_n$$

be the e.f of A

(207) $$A\varphi_n = a_n \varphi_n$$

follows

(208) $$BA\varphi_n = a_n B\varphi_n$$

Since

(209) $$AB = BA$$

(210) $$A(B\varphi_n) = a_n(B\varphi_n)$$

hence

(211) $$B\varphi_n \propto \varphi_n = b_n\varphi_n$$

or φ_n is e.f of B also

When there is degeneracy The e.f. can be chosen in such a way as to overlap

Three operators commute
$AB = BA \quad AC = CA \quad BC = CB$

See page 66

Angular momentum of a mass point

(212) $$\begin{cases} M_x = y p_z - z p_y \\ M_y = z p_x - x p_z \\ M_z = x p_y - y p_x \end{cases}$$

$\cancel{\frac{h}{2\pi i}(z p_y - y p_z)}$,

(213) $$M^2 = M_x^2 + M_y^2 + M_z^2$$

Commutation rules

(214) $$M_y M_z - M_z M_y = -\frac{h}{2\pi i} M_x$$

or

(215) $$[\vec{M} \times \vec{M}] = -\frac{h}{2\pi i}\vec{M}$$

(216) $$M^2 M_x - M_x M^2 = 0$$

Eigenvalues of

(217) M_z are $\frac{h}{2\pi} \underline{m}$ $\quad m = \text{integer}$

34 $-x\left(z p_z + \frac{h}{2\pi i}\right) p_x$ $\quad$ $\cancel{\Lambda + r^2\frac{\partial^2}{\partial r^2} + \frac{2r\,\partial}{\partial r} = r^2\Delta}$ $\quad$ $\cancel{x p_x x p_x}$

$\cancel{x\left(x p_x + \frac{h}{2\pi i}\right) p_x}$

(218) $$M^2 = (x p_z - z p_x)^2 + (y p_x - x p_y)^2 + (z p_y - y p_z)^2 =$$

$$= x^2 p_z^2 + z^2 p_x^2 + y^2 p_x^2 + x^2 p_y^2 + z^2 p_y^2 + y^2 p_z^2 + x^2 p_x^2 + y^2 p_y^2 + z^2 p_z^2$$

$$- x^2 p_x^2 - y^2 p_y^2 - z^2 p_z^2 - 2xz\,p_x p_z - 2xy\,p_x p_y - 2yz\,p_y p_z -$$

$\cancel{r\frac{\partial}{\partial r} r \frac{\partial}{\partial r}}$

$$- \frac{2h}{2\pi i}\left(x p_x + y p_y + z p_z\right)$$

$\cancel{r\left(r\frac{\partial}{\partial r} + 1\right)\frac{\partial}{\partial r}}$

$$= (x^2 + y^2 + z^2)(p_x^2 + p_y^2 + p_z^2) - (x p_x + y p_y + z p_z)^2 -$$

$$- \frac{h}{2\pi i}\left(x p_x + y p_y + z p_z\right)$$

(219) $$M^2 = -\frac{h^2}{4\pi^2}\left[r^2\Delta - r^2\frac{\partial^2}{\partial r^2} - 2r\frac{\partial}{\partial r}\right]$$

(220) $$r^2\Delta = \Lambda + r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r}$$

(221) $$M^2 = -\frac{h^2}{4\pi^2}\Lambda$$

Since e.v of Λ are $-l(l+1)$, e.v. of M are

$$\frac{h^2}{4\pi^2} l(l+1)$$

Eigenfun for

(222) $$\begin{cases} M_z = \frac{h}{2\pi} m \\ M^2 = \left(\frac{h}{2\pi}\right)^2 l(l+1) \end{cases}$$

are

(223) $$f(r)\, Y_{l,m}(\theta, \varphi)$$

Perturbation theory

~~Perturbation theory~~

~~Transformation theory~~

~~Instead of~~

~~Use the coefficients~~

(224) $$(H_0 + \mathcal{H})\, u = w\, u$$

(225) $$H_0\, u_n^0 = w_n^0\, u_n$$

(226) $$(H_0 + \lambda \mathcal{H})\left(u_i^{(0)} + \lambda u_i^{(1)} + \lambda^2 u_i^{(2)} + \dots\right) = \left(w_i^{(0)} + \lambda w_i^{(1)} + \dots\right)\left(u_i^{(0)} + \lambda u_i^{(1)} + \dots\right)$$

(227) $$\left\{\begin{aligned} & H_0 u_i^{(0)} = w_i^{(0)} u_i^{(0)} \\ & H_0 u_i^{(1)} - w_i^{(0)} u_i^{(1)} - w_i^{(1)} u_i^{(0)} = -\mathcal{H} u_i^{(0)} \\ & H_0 u_i^{(2)} - w_i^{(0)} u_i^{(2)} - w_i^{(2)} u_i^{(0)} = -\mathcal{H} u_i^{(1)} + w_i^{(1)} u_i^{(1)} \\ & \text{— — — —} \end{aligned}\right.$$

(228) $$u_i^{(1)} = \sum_j{}' c_{ij}^{(1)} u_j^{(0)} \;;\quad u_i^{(2)} = \sum_j{}' c_{ij}^{(2)} u_j^{(0)} \;; \dots$$

(229) $$\left\{\begin{aligned} & \sum_j{}' c_{ij}^{(1)} \left(w_j^{(0)} - w_i^{(0)}\right) u_j^{(0)} - w_i^{(1)} u_i^{(0)} = -\mathcal{H} u_i^{(0)} \\ & \sum_j{}' c_{ij}^{(2)} \left(w_j^{(0)} - w_i^{(0)}\right) u_j^{(0)} - w_i^{(2)} u_i^{(0)} = -\mathcal{H} u_i^{(1)} + w_i^{(1)} u_i^{(1)} \\ & \text{— — — —} \end{aligned}\right.$$

Put

(230) $$\mathcal{H}_{rs} = \int u_r^{(0)*}\, \mathcal{H}\, u_s^{(0)}\, d\tau$$

36

$$c_{ij}^{(2)} = \sum_l{}' \frac{\mathcal{H}_{jl}\,\mathcal{H}_{li}}{(W_i - W_l)(W_i - W_j)} - \frac{\mathcal{H}_{ji}\,\mathcal{H}_{ii}}{(W_i - W_j)^2}$$

(231) $$w_i^{(1)} = \mathcal{H}_{ii}$$

(232) $$c_{ij}^{(1)} = \frac{\mathcal{H}_{ji}}{w_i^{(0)} - w_j^{(0)}}$$

(233) $$w_i^{(2)} = \sum_j{}' \frac{\mathcal{H}_{ij}\,\mathcal{H}_{ji}}{w_i^{(0)} - w_j^{(0)}} = \sum_j{}' \frac{|\mathcal{H}_{ij}|^2}{w_i^{(0)} - w_j^{(0)}}$$

Example

Oscillator perturbed by constant force F

$$\mathcal{H} = -Fx$$

First order perturbation vanishes

$$w_0^{(2)} = -\frac{|\mathcal{H}_{01}|^2}{h\nu}$$

$$u_0^{(0)}(\xi) = N_0\, e^{-\frac{\xi^2}{2}}$$

$$u_1^{(0)}(\xi) = 2\xi N_1\, e^{-\xi^2/2}$$

$$\xi = 2\pi \sqrt{\frac{m\nu}{h}}\, x$$

$$N_0 = \left(\frac{4\pi m\nu}{h}\right)^{1/4} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$N_1 = \left(\frac{4\pi m\nu}{h}\right)^{1/4} \frac{1}{\sqrt{2}}$$

$$x_{01} = \int_{-\infty}^{\infty} N_0 N_1\, 2\xi\, e^{-\xi^2} x\, dx = \sqrt{\frac{h}{8\pi^2 m\nu}} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$x_{02} = x_{03} = \ldots = 0$$

Zeeman effect without spin

$$H = \frac{1}{2M}\left(p - \frac{e}{c}A\right)^2 + V(r)$$

$$= \frac{p^2}{2M} + V(r) - \frac{e}{Mc}\,p\cdot A + \text{quadratic term}$$

$$A_x = -\frac{H}{2}y$$
$$A_y = \frac{H}{2}x$$
$$A_z = 0$$

$$\text{Hamiltonian} = \frac{p^2}{2\cancel{M}} + U(r) + \frac{eH}{2\cancel{M}c}\left(x p_y - y p_x\right) =$$

Unperturbed

$$u_{n,l,m} = R_{nl}(r)\,Y_{l,m}(\vartheta,\varphi)$$

$$\left(\frac{p^2}{2m} + U_r\right)u_{nlm} = E_{nl}\,u_{nlm}$$

$$\text{Ham}\ u_{nlm} = E_{nl}\,u_{nlm} + \frac{e\hbar H m}{2MC}\,u_{nlm}$$

Discussion: Selection rule

$$m \rightarrow \begin{matrix} m+1 \\ m \\ m-1 \end{matrix}$$

Correspondence principle

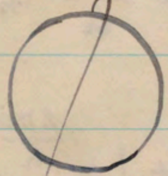
$$w_0^{(2)} = -\frac{F^2 h}{8\pi^2 m\nu \cdot h\nu} = -\frac{F^2}{8\pi^2 m\nu^2}$$

Direct proof:

$$H = \frac{p^2}{2m} + 2\pi^2 m \nu^2 x^2 - Fx$$

$$\cancel{= \frac{p^2}{2m} + 2\pi^2 m \nu^2 (x-\alpha)^2}$$

$$= \frac{p^2}{2m} + 2\pi^2 m \nu^2 \left(x - \frac{F}{4\pi^2 m\nu^2}\right)^2 - \frac{F^2}{8\pi^2 m\nu^2}$$



$$\frac{1}{\sqrt{2\pi}} e^{im\varphi}$$

$$\boxed{A\cos\varphi}$$

$$\cancel{\frac{1}{\sqrt{2\pi}} e^{im\varphi}}$$

$$\frac{1}{\sqrt{2\pi}} e^{-im\varphi}$$

$$\cancel{\frac{A}{2\pi} e^{-2im\varphi}\cos\varphi}$$

0

0

~~Hydrogen perturbed by constant field F~~

For perturbations with degeneracy see page 106

Time dependent perturbations

$$(H_0 + \mathcal{H})\psi = -\frac{h}{2\pi i}\frac{\partial \psi}{\partial t}$$

$\cancel{H_0 \psi_0}$ $\quad H_0 \cancel{u_n^{(0)} = W_n^{(0)} u_n^{(0)}}$

$\cancel{\psi = \sum c_n(t) u_n^{(0)}}$

$\cancel{c_n(t)\, w_n^{(0)}}$

$$\psi_n^{(0)} = u_n^{(0)} e^{-\frac{2\pi i}{h} W_n^{(0)} t}$$

$$\psi = \sum a_n(t)\, u_n^{(0)} e^{-\frac{2\pi i}{h} W_n^{(0)} t}$$

$$\dot{a}_n = -\frac{2\pi i}{h}\sum_m \mathcal{H}_{nm}(t)\, a_m(t)\, e^{\frac{2\pi i}{h}(W_n^\circ - W_m^\circ)t}$$

First approximation

$$a_n = a_n^\circ - \frac{2\pi i}{h}\sum_m a_m^\circ \int_0^t \mathcal{H}_{nm}(t)\, e^{\frac{2\pi i}{h}(W_n^\circ - W_m^\circ)t}\, dt$$

Probability of transition proportional to

$$|\mathcal{H}_{nm}|^2$$

$$\mathcal{H} = eAz \cos\omega t$$

$$\mathcal{H}_{nm} = eA z_{nm} \cos\omega t$$

$$\dot{a}_n \approx -\frac{ie}{2\hbar} A z_{nm} e^{i(\omega_{nm}-\omega)t}$$

$$a_n = +\frac{e}{2\hbar} A z_{nm} \frac{e^{i(\omega_{nm}-\omega)t} - 1}{\omega - \omega_{nm}}$$

$$|a_n|^2 = \frac{e^2 A^2 |z_{nm}|^2}{\hbar^2} \frac{\sin^2(\omega-\omega_{nm})\frac{t}{2}}{(\omega-\omega_{nm})^2}$$

$$\frac{A^2}{d\omega} \qquad \boxed{\frac{A^2}{d\omega}} d\omega \qquad \frac{cA^2}{8\pi} = dI$$

$$A^2 = \frac{8\pi}{c} \frac{dI}{d\omega}$$

$$|a_n|^2 = \frac{e^2}{\hbar^2} |z_{nm}|^2 \frac{8\pi}{c} \frac{dI}{d\omega} \frac{\pi t}{2}$$

prob. of transition $= \dfrac{4\pi^2 e^2}{c\hbar^2} |z_{nm}|^2 \dfrac{dI}{d\omega}$

Prob of transition in isotropic radiation of volume density $u(\omega)\,d\omega$

$$\frac{4}{3} \frac{\pi^2 e^2}{\hbar^2} |r_{nm}|^2 u(\omega)$$

Einsteins A, B

$$n_2 \left(A + Bu \right) = n_1 Bu$$

$$\frac{A}{Bu} + 1 = \frac{n_1}{n_2} = e^{\frac{\hbar\omega}{kT}}$$

$$\frac{A}{B} = u \times \left(e^{\frac{\hbar\omega}{kT}} - 1 \right)$$

$$d\omega\, u = \frac{\omega^2 d\omega}{\pi^2 c^3} \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}$$

$$\frac{A}{B} = \frac{\hbar\omega^3}{\pi^2 c^3} \qquad B = \frac{4\pi^2}{3} \frac{e^2}{\hbar^2} |r_{nm}|^2$$

$$A = \frac{4}{3} \frac{e^2 \omega^3}{\hbar c^3} |r_{nm}|^2 = \frac{1}{\tau}$$

Sinusoidal perturbation

$$\mathcal{H} = \mathcal{L} \cos 2\pi\nu t$$

Resonance

Perturbations produced by the radiation field

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi$$

Absorption and forced emission

Einstein's A's & B's $\qquad \dfrac{A}{B} = \dfrac{8\pi h \nu^3}{c^3} =$

Example Stark effect on Hydrogen

$n = 2 \qquad l = 0$

$$\cancel{R_{20}(r)}\qquad R_{20} = \frac{1}{\sqrt{2}\, a^{3/2}} e^{-\frac{r}{2a}} \left(1 - \frac{r}{2a}\right)$$

$$R_{21} = \frac{1}{2\sqrt{6}\, a^{3/2}} e^{-\frac{r}{2a}} \frac{r}{a}$$

taking unity of length $a = \dfrac{h^2}{4\pi^2 m e^2}$

normalized

$$\psi_{20} = \frac{1}{2\sqrt{2\pi}} \left(1 - \frac{r}{2}\right) e^{-\frac{r}{2}}$$

$$\psi_{21} = \frac{1}{2\sqrt{6}} e^{-\frac{r}{2}} r \begin{cases} \sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{i\varphi} \\ \sqrt{\frac{3}{4\pi}} \cos\vartheta \\ \sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{-i\varphi} \end{cases}$$

$$\psi_{10} = \frac{1}{\sqrt{\pi}} e^{-r}$$

Einstein's A's & B's

$$\frac{n_2}{n_1} = e^{-\frac{h\nu}{kT}}$$

$$n_1 B u_\nu = n_2 (A + B' u_\nu)$$

$$u_\nu \left[B e^{\frac{h\nu}{kT}} - B' \right] = A$$

$$u_\nu = \frac{A}{B\left\{ e^{\frac{h\nu}{kT}} - \frac{B'}{B} \right\}} = \frac{8\pi h\nu^3/c^3}{e^{\frac{h\nu}{kT}} - 1}$$

$$\begin{cases} B' = B \\ \dfrac{A}{B} = \dfrac{8\pi h\nu^3}{c^3} \end{cases}$$

Absorption of radiation

~~Field~~ Perturbing potential

$$e x \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi n}{T} t + \beta_n\right)$$

$$\dot{a}_2 = -\frac{2\pi i}{h} e x_{12} \sum A_n \cos\left(\frac{2\pi n}{T} t + \beta_n\right) e^{2\pi i \nu t}$$

$$W_2 - W_1 = h\nu$$

$$\dot{a}_2 = -\frac{2\pi i}{h} e x_{12} \sum_n \frac{A_n}{2} e^{-i\beta_n} e^{+2\pi i\left(\nu - \frac{n}{T}\right)t}$$

$$a_2 = -\frac{e\,x_{12}}{2h}\sum_n A_n e^{-i\beta_n} \frac{e^{2\pi i\left(\nu-\frac{n}{T}\right)t}-1}{\nu-\frac{n}{T}}$$

$$|a_2|^2 = \frac{e^2|x_{12}|^2}{\cancel{4}\,h^2}\sum_n A_n^2 \frac{\sin^2\pi t\left(\frac{n}{T}-\nu\right)}{\left(\frac{n}{T}-\nu\right)^2} + \sum_{nm}\cdots$$

$$\overline{|a_2|^2} = \frac{e^2|x_{12}|^2}{h^2}\overline{A_\nu^2}\int_{-\infty}^{\infty}\frac{\sin^2\pi t\left(\frac{n}{T}-\nu\right)}{\left(\frac{n}{T}-\nu\right)^2}\,dn$$

$$u_\nu\,d\nu = \frac{1}{8\pi}\left(\overline{E_x^2}+\overline{E_y^2}+\overline{E_z^2}+\overline{H_x^2}+\overline{H_y^2}+\overline{H_z^2}\right)_\nu(d\nu)$$

$$= \frac{3}{4\pi}\overline{E_x^2}(d\nu) = \frac{3}{4\pi}\frac{\overline{A_\nu^2}}{2}T\,d\nu$$

$$u_n = \frac{3}{8\pi}\overline{A_\nu^2}\,T$$

$$\overline{|a_2|^2} = \frac{e^2|x_{12}|^2}{h^2}\pi^2 t\,T\,\overline{A_\nu^2}$$

$$= \frac{8\pi^3}{3}\frac{e^2|x_{12}|^2}{h^2}u_\nu t$$

$$B = \frac{8\pi^3}{3}\frac{e^2}{h^2}\left\{|x_{12}|^2+|y_{12}|^2+|z_{12}|^2\right\}$$

$$A = \frac{1}{\tau} = \frac{64\pi^4 e^2\nu^3}{3c^3h}\left[|x_{12}|^2+|y_{12}|^2+|z_{12}|^2\right]$$

Application to point in central field of force

Eigenfunctions

$$R_n(r)\, Y_{lm}(\vartheta, \varphi) = u_{nlm}$$

Interpretation of q. n. $\underline{m}$ & $\underline{l}$

$$M_z = \frac{h}{2\pi i}\frac{\partial}{\partial \varphi}$$

$$M^2 = -\frac{h^2}{4\pi^2}\Lambda$$

$$M^2 u_{nlm} = -\frac{h^2}{4\pi^2} R_n(r)\, \Lambda\, Y_{lm} =$$

$$= \frac{h^2}{4\pi^2}\, l(l+1)\, u_{nlm}$$

thus

$$M^2 = \left(\frac{h}{2\pi}\right)^2 l(l+1)$$

or total momentum is $\sqrt{l(l+1)}$ in units $\hbar$.

$$M_z u_{nlm} = m\, u_{nlm}$$

or

$\underline{z}$ component of ang momentum is $\underline{m}$ in units $\hbar$.

$$Y_{11} Y_{l\,m-1} = \sqrt{\frac{3}{8\pi}} \sqrt{\frac{(l+m)(l+1+m)}{(2l+1)(2l+3)}}\, Y_{l+1,m} - \sqrt{\frac{3}{8\pi}} \sqrt{\frac{(l-m)(l+1-m)}{(2l+1)(2l-1)}}\, Y_{l-1,m}$$

$$Y_{10} Y_{lm} = \sqrt{\frac{3}{4\pi}} \sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}\, Y_{l+1,m} + \sqrt{\frac{3}{4\pi}} \sqrt{\frac{l^2 - m^2}{(2l+1)(2l-1)}}\, Y_{l-1,m}$$

$$Y_{1,-1} Y_{l\,m+1} = \sqrt{\frac{3}{8\pi}} \sqrt{\frac{(l-m)(l+1-m)}{(2l+1)(2l+3)}}\, Y_{l+1,m} - \sqrt{\frac{3}{8\pi}} \sqrt{\frac{(l+m)(l+1+m)}{(2l+1)(2l-1)}}\, Y_{l-1,m}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\vartheta \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{i\varphi}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{-i\varphi}$$

$$\int \cos\vartheta\, Y_{lm} Y^*_{l+1\,m}\, d\omega = \sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}$$

$$\int \cos\vartheta\, Y_{lm} Y^*_{l-1\,m}\, d\omega = \sqrt{\frac{l^2 - m^2}{(2l+1)(2l-1)}}$$

$$\int \sin\vartheta\, e^{\pm i\varphi}\, Y_{l\,m\mp 1} Y^*_{l+1\,m}\, d\omega = \mp \sqrt{\frac{(l\pm m)(l+1\pm m)}{(2l+1)(2l+3)}}$$

$$\int \sin\vartheta\, e^{\pm i\varphi}\, Y_{l,m\mp 1} Y^*_{l-1\,m}\, d\omega = \pm \sqrt{\frac{(l\mp m)(l+1\mp m)}{(2l+1)(2l-1}}$$

Selection rules

States

$$u_{nlm} = R_n(r)\, P_{lm}(\cos\vartheta)\, e^{im\varphi}$$

$$u_{n'l'm'} = R_{n'l'}(r)\, P_{l'm'}(\cos\vartheta)\, e^{im'\varphi}.$$

$$x = r \sin\vartheta \cos\varphi$$

$$y = r \cancel{\cos}\,\vartheta \sin\varphi$$

$$z = r\cos\vartheta$$

$$x_{nlm,\,n'l'm'} = \int_0^\infty r^3 R_{n'l'}(r) R_n(r)\, dr \times$$

y

$$\times \int_0^\infty \sin^2\vartheta\, P_{l'm'}(\cos\vartheta)\, P_{lm}(\cos\vartheta)\, d\vartheta \times$$

$$\times \int_0^\infty \begin{matrix}\cos\varphi \\ \sin\varphi\end{matrix}\; e^{i(m-m')\varphi}\, d\varphi$$

$$z_{nlm,\,n'l'm'} = \int_0^\infty r^3 R_{n'} R_n\, dr \times \int_0^\infty \sin\vartheta\cos\vartheta\, P_{l'm'} P_{lm}\, d\vartheta \times$$

$$\times \int e^{i(m-m')\varphi}\, d\varphi$$

$$\Delta m = 0, \pm 1$$

$$\Delta l = \pm 1$$

Lifetime of u_{210} of Hydrogen

$$z_{100,\,210} = \frac{2^7\sqrt{2}}{3^5}\,a = 0.75\,a$$

Theorem

v_l

be a homog polinomial of $x\ y\ z$ degree l

Then

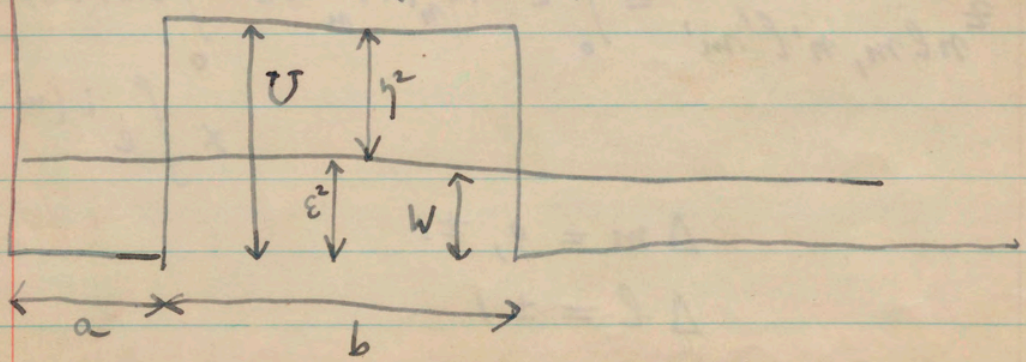
$$\cancel{v_l = \int \sum}$$

$$\cancel{v_l = r^l \sum^{[l/2]} Y}$$

$$v_l = r^l \sum_0^{[l/2]} \sum_{m=-(l-2n)}^{l-2n} c_{l-2n,\,m}\ Y_{l-2n,\,m}$$

~~Transparency of a potential wall~~

Transparency of a potential wall



$$\psi'' + \frac{8\pi^2 m}{h^2}(E-U)\psi \qquad e^-$$

$$4A^2 = e^{2b\eta}\left(\sin\varepsilon a + \frac{\varepsilon}{\eta}\cos\varepsilon a\right)^2\left(1+\frac{\eta^2}{\varepsilon^2}\right) +$$

$$+ 2\left(\sin^2\varepsilon a - \frac{\varepsilon^2}{\eta^2}\cos^2\varepsilon a\right)\left(1-\frac{\eta^2}{\varepsilon^2}\right)$$

$$+ e^{-2b\eta}\left(\sin\varepsilon a - \frac{\varepsilon}{\eta}\cos\varepsilon a\right)\left(1+\frac{\eta^2}{\varepsilon^2}\right)$$

$\}$

$$\cancel{= p'^2 w^2 e^{2by} + 2 P}$$

$$\cancel{e^{2by}\left(\frac{w-w_0}{\varepsilon}\right)^2 + e^{-2b}}$$

Gamow theory of α particle emission

~~Life time~~ α

$$\frac{1}{\tau} \sim \frac{v}{a} e^{-2\sqrt{\frac{8\pi^2 m}{h^2}(U-E)}\, b}$$

Gamow Pot barrier

$$\frac{1}{\tau} = \frac{v_i}{\rho} e^{-\frac{8\pi e^2}{h}\frac{2Z}{v}\left(\arccos\frac{1}{\sqrt{\sigma}} - \sqrt{\frac{1}{\sigma}-\frac{1}{\sigma^2}}\right)}$$

$$\sigma = \frac{2Ze^2}{E_0\rho}$$

U	4.09	$4.4\,10^9$ a
Ra	4.79	1550 a
Rn	5.49	3.82 d
RaC'	7.68	10^{-5} sec

U 4.09 Mev

~~Ra~~ s

Spin of the electron (Pauli)

$M_x \neq$

$M_y \neq$

$M_z =$

$$M_y M_z - M_z M_y = -\frac{h}{2\pi i} M_x$$

$$\mu_x = \frac{h}{4\pi}\sigma_x$$

$$\mu_y = \frac{h}{4\pi}\sigma_y$$

$$\mu_z = \frac{h}{4\pi}\sigma_z$$

$$\sigma_y \sigma_z - \sigma_z \sigma_y = 2i\,\sigma_x$$

$$\sigma_x \sigma_y = -\sigma_y \sigma_x$$

$$\sigma_y \sigma_z = i\,\sigma_x$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} \psi \\ 0 \end{vmatrix} = \begin{vmatrix} a\psi \\ b\psi \end{vmatrix} = \begin{vmatrix} \psi \\ 0 \end{vmatrix}$$

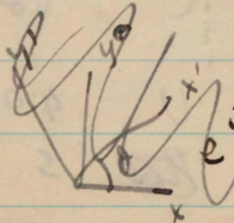
$a = 1 \qquad d = -1$

$c = 0 \qquad b = 0$

$$\sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \qquad \sigma_x = \begin{vmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{vmatrix} \qquad \sigma_y = \begin{vmatrix} 0 & e^{i\beta} \\ e^{-i\beta} & 0 \end{vmatrix}$$

$$\alpha - \beta = \frac{\pi}{2} + n\pi$$

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \qquad \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}$$



$$e^{i\alpha}\cos\theta - i\,e^{i\alpha}\sin\theta$$

Generalities on electron spin

Fourth coordinate ξ

$$\psi(x, y, z, \xi)$$

ξ is two valued (take $\xi = \pm 1$

$$\psi(x, y, z, \xi) = \begin{cases} \psi_+(xyz) \\ \psi_-(xyz) \end{cases}$$

Spin operators $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

Search for three operators

$$\sigma_x, \sigma_y, \sigma_z$$

to represent the 3 components of the spin vector Normalise them so that e.v. are ± 1. Hence

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

Also with α, β, γ arbitrary direction cosines

$$(\alpha \sigma_x + \beta \sigma_y + \gamma \sigma_z)^2 = 1$$

From this: anticommutation

$$\sigma_y \sigma_z + \sigma_z \sigma_y = 0, \ldots$$

By proper choice of the <u>base</u> one can always assume

$$\sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

Follows) $\sigma_x = \begin{vmatrix} a & b \\ b^* & c \end{vmatrix}$ $\sigma_x \sigma_z + \sigma_z \sigma_x = 0$

$$0 = \begin{vmatrix} a & -b \\ b^* & -c \end{vmatrix} + \begin{vmatrix} a & b \\ -b^* & -c \end{vmatrix} = 2\begin{vmatrix} a & 0 \\ 0 & -c \end{vmatrix}$$

Hence $a = c = 0$

$$\sigma_x = \begin{vmatrix} 0 & b \\ b^* & 0 \end{vmatrix}$$ From $\sigma_x^2 = 1$

$bb^* = 1$ $b = e^{i\alpha}$

$$\sigma_x = \begin{vmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{vmatrix}$$

~~Now $\sigma_y \sigma_x + \sigma_x \sigma_y = 0$~~ $\sigma_y = \begin{vmatrix} 0 & e^{i\beta} \\ e^{-i\beta} & 0 \end{vmatrix}$

$$\sigma_x \sigma_y = \begin{vmatrix} e^{i(\alpha-\beta)} & 0 \\ 0 & e^{-i(\alpha-\beta)} \end{vmatrix} \quad \sigma_y \sigma_x = \begin{vmatrix} e^{-i(\alpha-\beta)} & 0 \\ 0 & e^{i(\alpha-\beta)} \end{vmatrix}$$

Since $\sigma_x \sigma_y = -\sigma_y \sigma_x$ follows $e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)} = 2\cos(\alpha-\beta) = 0$

~~$\alpha - \beta = \pi/2$~~ $\alpha - \beta = \pm \frac{\pi}{2}$ Hence $\sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$ σ_x

---

α can be taken equal to 0

β " " " " " $-\frac{\pi}{2}$

Proof: by transformation $\psi_+^{new} = e^{-i\alpha} \psi_+^{old}$

$\psi_-^{new} = \psi_-^{old}$

one gets

$$\sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \quad \sigma_x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad \sigma_y = \begin{vmatrix} 0 & \pm i \\ \mp i & 0 \end{vmatrix}$$

If upper sign change direction of vector and transform $\psi_+^{new} = -\psi_-^{old}$ $\psi_-^{new} = \psi_+^{old}$ Finally $\begin{vmatrix} 1 & \\ & -1 \end{vmatrix} \begin{vmatrix} & 1 \\ 1 & \end{vmatrix} \begin{vmatrix} & -i \\ i & \end{vmatrix}$

46 $\frac{2}{3}$

$$\sigma_x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad \sigma_y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \quad \sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

$$\sigma_y \sigma_z = \begin{vmatrix} 0 & i \\ i & 0 \end{vmatrix} = i\sigma_x \qquad \sigma_y\sigma_z - \sigma_z\sigma_y = 2i\,\sigma_x$$

$$\sigma \times \sigma = 2i\sigma$$

For angular momentum vectors

$$M \times M = -\frac{\hbar}{i} M$$

Suggests to take angular momentum associated to σ

$$a\sigma \qquad a\sigma \times a\sigma = -\frac{\hbar}{i} a\sigma$$

or $\cancel{a}$ $\quad 2ia^2\sigma = -\frac{\hbar}{i} a\sigma$

$\cancel{2ia^2\sigma = -\frac{\hbar}{i}a\sigma}$ $\quad \boxed{a = \frac{\hbar}{2}}$

Spin angular momentum $= \frac{\hbar}{2}\sigma$

Spin magn. moment $= \frac{e\hbar}{2mc}\sigma$ to fit expt data

or from Dirac theory

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 = \frac{p^2}{2m} - eV(r) + \frac{\mu_0 \hbar}{2mc}\frac{F(r)}{r}(l \cdot \sigma)$$

Unpert. e.f $\quad \begin{vmatrix} R_{nl}(r)\, Y_{lm}(\vartheta, \varphi) \\ 0 \end{vmatrix}$ and $\begin{vmatrix} 0 \\ R_{nl}\, Y_{lm} \end{vmatrix}$

$$(l_x + i l_y)\, Y_{lm} = -\sqrt{(l-m)(l+m+1)}\; Y_{l\,m+1}$$
$$(l_x - i l_y)\, Y_{lm} = -\sqrt{(l+m)(l-m+1)}\; Y_{l,m-1}$$
$$l_z Y_{lm} = m Y_{lm}$$

$$(l\cdot\sigma) = l_z\sigma_z + \tfrac{1}{2}(l_x+il_y)(\sigma_x-i\sigma_y) + \tfrac{1}{2}(l_x-il_y)(\sigma_x+i\sigma_y)$$

$$\frac{\sigma_x+i\sigma_y}{2} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$\frac{\sigma_x-i\sigma_y}{2} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$$

$$(l\cdot\sigma)\begin{vmatrix} Y_{lm} \\ 0 \end{vmatrix} = \begin{vmatrix} mY_{lm} \\ -\sqrt{(l-m)(l+m+1)}\;Y_{l\,m+1} \end{vmatrix}$$

$$(l\cdot\sigma)\begin{vmatrix} 0 \\ Y_{l,m+1} \end{vmatrix} = \begin{vmatrix} -\sqrt{(l+m+1)(l-m)}\;Y_{l,m} \\ -(m+1)\,Y_{l,m+1} \end{vmatrix}$$

Pert. matrix

$$\frac{\mu_0\hbar}{2mc}\left(\int\frac{F(r)}{r}R^2_{nl}(r)\,r^2dr\right)\begin{vmatrix} m & -\sqrt{(l-m)(l+m+1)} \\ -\sqrt{} & -(m+1) \end{vmatrix}$$

$$(m-x)(m+1+x) - (l-m)(l+1+m) = 0 \qquad x = \begin{cases} l & j = l+\frac{1}{2} \\ -(l+1) & j = l-\frac{1}{2} \end{cases}$$

$$\vec{l} + \tfrac{1}{2}\vec{\sigma} = \vec{j} \qquad j(j+1) = l(l+1) + \frac{3}{4} + (l\cdot\sigma)$$

~~Eigenfunctions~~

$$\text{Splitting} = (2l+1)\frac{\mu_0\hbar}{2mc}\int\frac{F(r)}{\cancel{r}}R^2_{nl}\,r\,dr$$

e.f's $\quad j = l+\frac{1}{2} \quad m' = m+\frac{1}{2} \quad l.\sigma = l$

$$R_{nl}(r)\begin{vmatrix} \sqrt{\frac{l+m+1}{2l+1}}\;Y_{lm} \\ -\sqrt{\frac{l-m}{2l+1}}\;Y_{l,m+1} \end{vmatrix} \qquad \frac{R_{nl}(r)}{\sqrt{(l-m)(2l+1)}}\begin{vmatrix} \sqrt{(l-m)(l+m+1)}\;Y_{l,m} \\ -(l-m)\,Y_{l,m+1} \end{vmatrix}$$

$$R_{nl}(r)\begin{vmatrix} \sqrt{\frac{l-m}{2l+1}}\;Y_{lm} \\ \sqrt{\frac{l+m+1}{2l+1}}\;Y_{l,m+1} \end{vmatrix}$$

50

Systems with several particles &

$$H = H_1 + H_2 + H_3 + \dots + H_n$$

~~wave~~ eigenfunctions are products

eigenvalues are sums

Systems with two identical particles

H is symmetrical

means

$$H\psi(x_1\, x_2) = \varphi(x_1\, x_2)$$

$$H\psi(x_2\, x_1) = \varphi(x_2\, x_1)$$

Follows: if

$$\psi(x_1, x_2) = \pm\psi(x_2\, x_1)$$

also

$$\varphi(x_1\, x_2) = \pm\varphi(x_2\, x_1)$$

Hence complete separation of sym & antisym states

Same holds for many identical particles

Pauli principle

If wavefunctions are antisym no two particles can be in one well def quantum state

Bose Einstein statistics

Where functions are simmetrical a quantum state is defined by the occupation numbers

Two particle system.

Assume
$$H = H^{(1)} + H^{(2)}$$
$$= \frac{p_1^2}{2m_1} + U_1(x_1) + \frac{p_2^2}{2m_2} + U_2(x_2)$$

$$u_i^{(1)}(x_1)\, u_k^{(2)}(x_2)$$

Show completeness of system

Effect of interaction
$$H^{(12)}$$

Matrix elements
$$H^{(12)}_{ik,\,lm} = \int u_i^{(1)*}(x_1)\, u_k^{(2)*}(x_2)\, H^{(12)}\, u_l^{(1)}(x_1)\, u_m^{(2)}(x_2)\, dx_1 dx_2$$

So that in general
$$u(x_1, x_2)$$

Identical particles
$u(x_1, x_2)$ is same state as $u(x_2 x_1)$

Hence
$$u(x_1\, x_2) = \pm\, u(x_2 x_1)$$

Symmetric Hamiltonian does not change symmetry charachter – Hence charachter as initial –

Bose Einstein & Pauli particles

Spin and antisymmetry

$$\cancel{u(x)}\ u_\alpha(x_1\xi_1)\,u_\beta(x_2\xi_2) - u_\alpha(x_2\xi_2)\,u_\beta(x_1\xi_1)$$

He spectrum

$1s^2$	$^1S_0$	198298
$1s2s$	$^3S_1$	38455
	$^1S_0$	32033
$1s2p$	$3P$	

mercury

$6s^2$	84178
$6s6p$	46536
	44769
	40138

$H = H_1 + H_2 + \frac{e^2}{r_{12}}$ ① 50

$$H_1 = \frac{p_1^2}{2m} + U(x_1) \qquad H_2 = \frac{p_2^2}{2m} + U(x_2)$$

$u_1(x) \ldots \quad u_n(x) \ \ldots \quad u_m(x) \ldots$

$E_1 \qquad E_n \qquad E_m$

$\alpha(\xi)$ define $\quad \alpha(1) = 1 \quad \alpha(-1) = 0$

$\beta(\xi) \qquad\qquad \beta(1) = 0 \quad \beta(-1) = 1$

---

one electron states $\underline{n}$

$u_n(x)\alpha(\xi)$ or $u_n(x)\beta(\xi)$

---

two electron states n, m

$\Psi_1 \quad \frac{1}{\sqrt{2}}\left[u_n(x_1)\alpha(\xi_1)u_m(x_2)\alpha(\xi_2) - u_m(x_1)\alpha(\xi_1)u_n(x_2)\alpha(\xi_2)\right]$

$\Psi_2 \quad \frac{1}{\sqrt{2}}\left[u_n(x_1)\alpha(\xi_1)u_m(x_2)\beta(\xi_2) - u_m(x_1)\beta(\xi_1)u_n(x_2)\alpha(\xi_2)\right]$

$\Psi_3 \quad \frac{1}{\sqrt{2}}\left[\quad \beta \qquad \alpha \qquad \alpha \qquad \beta\right.$

$\Psi_4 \quad \frac{1}{\sqrt{2}}\left[\quad \beta \qquad \beta \qquad \beta \qquad \beta\right.$

---

normalization $1/\sqrt{2}$

$$|u_n(x_1)|^2|u_m(x_2)|^2\alpha^2(\xi_1)\beta^2(\xi_2) +$$

$$|u_m(x_1)|^2|u_n(x_2)|^2\beta^2(\xi_1)\beta^2(\xi_2) -$$

$$- \underline{u_n(x_1)u_m^*(x_1)}\ \underline{u_m(x_2)u_n^*(x_2)}\ \underline{\alpha(\xi_1)\beta(\xi_1)}\ \underline{\alpha(\xi_2)\beta(\xi_2)}$$

$-$ comp conj

$$\int \sum_{\xi_1,\xi_2}\iint d^3x_1\, d^3x_2 = \left(\int |u_n(x_1)|^2 dx_1\right)\left(\int |u_m(x_2)|^2 dx_2\right)\left(\sum_{\xi_1=\pm 1}\alpha^2(\xi_1)\right)\left(\sum_{\pm 1}\beta^2(\xi_2)\right)$$

Perturb. matrix 4×4 elements

Prove that on one or two examples

$$\begin{array}{c|c|cc|c|} & 1 & 2 & 3 & 4 \\ \hline 1 & \bullet & 0 & & 0 \\ \hline 2 & & \bullet & \bullet & \\ 3 & 0 & \bullet & \bullet & 0 \\ \hline 4 & 0 & 0 & & \bullet \\ \hline \end{array}$$

so six matrix elements

$$\mathcal{H}_{11} = \frac{e^2}{2}\iint \sum_{\substack{\xi_1 \\ \pm 1}}\sum_{\substack{\xi_2 \\ \pm 1}} \frac{dx_1 dx_2}{r_{12}}\Big(u_n^*(x_1)\, u_m^*(x_2)\, \alpha(\xi_1)\alpha(\xi_2)\, u_n(x_1)\, u_m(x_2)\alpha(\xi_1)\alpha(\xi_2)$$

$$- u_n^*(x_1)\, u_m^*(x_2)\alpha(\xi_1)\alpha(\xi_2)\, u_m(x_1)\, u_n(x_2)\alpha(\xi_1)\alpha(\xi_2)$$

$$- u_m^*(x_1)\, u_n^*(x_2)\, \alpha(\xi_1)\alpha(\xi_2)\, u_n(x_1)\, u_m(x_2)\alpha\alpha$$

$$+ u_m^*(x_1)\, u_n^*(x_2)\, \alpha \quad \alpha \;\; u_m(x_1) u_n(x_2)\alpha\alpha\Big)$$

Sum part comes out and is $=1$

~~Put~~ 1st & 4th line equal their sum is

$$A = e^2 \iint \frac{|u_n(x_1)|^2\, |u_m(x_2)|^2}{r_{12}}\, dx_1 dx_2$$

2nd & 3rd lines equal their sum

$$-B = e^2 \iint \frac{u_n^*(x_1)\, u_m^*(x_2)\, u_m(x_1)\, u_n(x_2)}{r_{12}}\, dx_1 dx_2 \quad \text{is real}$$

Total $\mathcal{H}_{11} = A - B$

Similar $\mathcal{H}_{44} = A - B$

$\mathcal{H}_{22} = \mathcal{H}_{33} = A$

$\mathcal{H}_{23}$ $\quad \mathcal{H}_{23} = \mathcal{H}_{32}^* = -B$

$$\begin{vmatrix} A-B & 0 & 0 & 0 \\ 0 & A, & -B & 0 \\ 0 & -B & A & 0 \\ 0 & 0 & 0 & A-B \end{vmatrix}$$

Find e.v. solving separately ② 50

$|A-B|$ has e.v $\varepsilon = A-B$

$$\begin{vmatrix} A-\varepsilon & -B \\ -B & A-\varepsilon \end{vmatrix} = 0 \qquad \varepsilon^2 - 2A\varepsilon + A^2 - B^2 = 0$$

$$A \pm \sqrt{A^2 - A^2 + B^2} = A \pm \sqrt{B^2} = A \pm B$$

$$\varepsilon = \begin{cases} A-B \\ A+B \end{cases}$$

$|A-B|$ $\qquad \varepsilon = A-B$

three times $A-B$
once $A+B$

Find e.f's to $\underline{\varepsilon = A-B}$

$$\psi_1 = \frac{1}{\sqrt{2}}\left\{u_n(x_1)\,u_m(x_2) - u_m(x_1)\,u_n(x_2)\right\}\alpha(\xi_1)\,\alpha(\xi_2)$$

$$\psi_4 = \frac{1}{\sqrt{2}}\left\{ \qquad \right\}\beta(\xi_1)\,\beta(\xi_2)$$

also substitute $\varepsilon = A-B$ in sec. determinant

$$\begin{vmatrix} B & -B \\ -B & B \end{vmatrix} = 0$$ minors are coefficients
they are equal
hence sum $\dfrac{\psi_2 + \psi_3}{\sqrt{2}}$

$$\frac{\psi_2 + \psi_3}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left\{u_n(x_1)\,u_m(x_2) - u_m(x_1)\,u_n(x_2)\right\} \frac{\alpha(\xi_1)\,\beta(\xi_2) + \alpha(\xi_2)\,\beta(\xi_1)}{\sqrt{2}}$$

corresp to $\varepsilon = A+B$ $\begin{vmatrix} -B & -B \\ -B & -B \end{vmatrix}$ minors $-B, +B$ or $1, -1$

$\frac{1}{\sqrt{2}}(\psi_2 - \psi_3)$

$$\frac{\psi_2-\psi_3}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left\{u_n(x_1)\,u_m(x_2) + u_m(x_2)\,u_n(x_1)\right\} \frac{\alpha(\xi_1)\beta(\xi_2) - \beta(\xi_1)\alpha(\xi_2)}{\sqrt{2}}$$

Notice to $\varepsilon = A - B$
e.f. orbital part antisym
spin part sym

---

to $\varepsilon = A+B$
orbital anti
spin symmetric

Consider spin functions only

symmetric	antisymmetric
$\alpha(\xi_1)\,\alpha(\xi_2)$	$\frac{\alpha(\xi_1)\beta(\xi_2) - \beta(\xi_1)\alpha(\xi_2)}{\sqrt{2}}$
$\frac{\alpha(\xi_1)\beta(\xi_2) + \alpha(\xi_2)\beta(\xi_1)}{\sqrt{2}}$	
$\beta(\xi_1)\,\beta(\xi_2)$	

Statement: For sym states $S = s_1 + s_2 = 1$ (in units $\hbar$) or $S^2 = 1\times 2 = 2$
For antis state $S = 0$ or $S^2 = 0$

$\vec{s}_1 = \frac{1}{2}\vec{\sigma}_1 \qquad \vec{s}_2 = \frac{1}{2}\sigma_2$

$$S^2 = \frac{1}{4}\left[(\sigma_{1x}+\sigma_{2x})^2 + (\sigma_{1y}+\sigma_{2y})^2 + (\sigma_{2z}+\sigma_{2z})^2\right] = \frac{1}{4}\left(6 + 2(\sigma_{1x}\sigma_{2x} + \dots)\right) =$$

$$S^2 = \frac{3}{2} + \frac{1}{2}\left(\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z}\right)$$

$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$: $\sigma_y \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}$ $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$ 50

(3)

$$\sigma_x \alpha(\xi) = \beta(\xi) \qquad \sigma_y \alpha = i\beta \qquad \sigma_z \alpha = \alpha$$

$$\sigma_x \beta = \alpha \qquad \sigma_y \beta = -i\alpha \qquad \sigma_z \beta = -\beta$$

$$(\sigma_1 \cdot \sigma_2)\, \alpha\alpha = \beta\beta + i^2 \beta\beta + \alpha\alpha = \alpha\alpha$$

$$(\sigma_1 \sigma_2) \frac{\alpha\beta + \beta\alpha}{\sqrt{2}} = \frac{(\beta\alpha - i^2\beta\alpha - \alpha\beta) + (\alpha\beta - i^2\alpha\beta - \beta\alpha)}{\sqrt{2}} = \frac{\beta\alpha + \alpha\beta}{\sqrt{2}}$$

$$(\sigma_1 \sigma_2)\, \beta\beta = \beta\beta$$

for symm fs $(\sigma_1 \cdot \sigma_2) = 1 \qquad S^2 = \frac{3}{2} + \frac{1}{2} = 2$

---

antis

$$(\sigma_1 \cdot \sigma_2) \frac{\alpha\beta - \beta\alpha}{\sqrt{2}} = \frac{(\quad) - (\quad)}{\sqrt{2}} = 3\, \frac{\beta\alpha - \alpha\beta}{\sqrt{2}}$$

$$(\sigma_1 \sigma_2) = -3 \qquad S^2 = \frac{3}{2} - \frac{3}{2} = 0$$

$$S_z = \frac{1}{2}(\sigma_{1z} + \sigma_{2z}) \begin{Bmatrix} \alpha\alpha \\ \frac{\alpha\beta + \beta\alpha}{\sqrt{2}} \\ \beta\beta \end{Bmatrix} = \begin{matrix} 1 \times \alpha\alpha \\ 0 \times \frac{\quad}{\sqrt{2}} \\ -1 \times \beta\beta \end{matrix}$$

$$S_z \frac{\alpha\beta - \beta\alpha}{\sqrt{2}} = 0$$

Hence — Spin symm states are said to have total spin 1 (Spins ∥)

Spin antis states have total spin 0 (Spins antiparallel)

Case $n = m$ and general discussion of the He spectrum

Three states, two particles

~~Boltzmann~~

State	Bzm	BE	Pauli
(2 0 0)	$\boxed{ab}\boxed{\ }\boxed{\ }$	$\boxed{aa}\boxed{\ }\boxed{\ }$	—
(0 2 0)	do		—
(0 0 2)			—
(0 1 1)	$\left\{\begin{array}{l}\boxed{\ }\boxed{a}\boxed{b}\\ \boxed{\ }\boxed{b}\boxed{a}\end{array}\right.$	$\boxed{\ }\boxed{a}\boxed{a}$	$\boxed{\ }\boxed{a}\boxed{a}$

Pauli principle with spin

$$\psi(x_1 \zeta_1, x_2 \zeta_2) = -\psi(x_2 \zeta_2\ x_1 \zeta_1)$$

$$\psi_{++}(x_1\ x_2) = -\psi_{++}(x_2\ x_1)$$

$$\psi_{--}(x_1\ x_2) = -\psi_{--}(x_1\ x_2)$$

$$\psi_{+-}(x_1\ x_2) = -\psi_{-+}(x_2\ x_1)$$

Suppose H contains only operators acting on the coordinates; then e.v problem gives

$$H\psi_{++}(x_1, x_2) = E\psi_{++}(x_1\ x_2)$$

$$H\psi_{+-}(x_1, x_2) = E\psi_{+-}(x_1 x_2)$$

$$H\psi_{-+}(x_1, x_2) = E\psi_{-+}(x_1 x_2)$$

$$H\psi_{--}(x_1, x_2) = E\psi_{--}(x_1, x_2)$$

52

Let E_a & E_s be two (non deg) eigenvalues of H with e.f. $u_a(x_1 x_2) = -u_a(x_2 x_1)$ and $u_s(x_1 x_2) = u_s(x_2 x_1)$

Without symmetry limitations one would expect:

for $H = E_a$

quadruple degeneracy

$H=E_a$
$$\begin{cases} u_a(x_1, x_2) = \psi_{++} \ ; \ \psi_{+-} = \psi_{-+} = \psi_{--} = 0 \\ u_a = \psi_{+-} \ ; \ \psi_{++} = \psi_{-+} = \psi_{--} = 0 \\ u_a = \psi_{-+} \quad \text{others zero} \\ u_a = \psi_{--} \quad \text{others zero} \end{cases}$$

good combinations for electrons

$u_a = \psi_{++}$

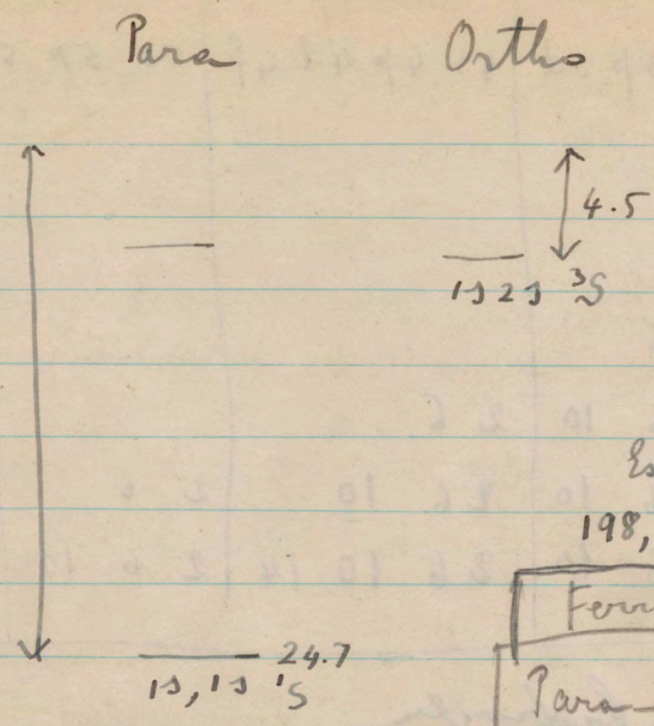
$\frac{1}{\sqrt{2}} u_a = \frac{\psi_{+-}}{\sqrt{2}}$;

$u_a = \psi_{--}$

$H=E_s$
$$\begin{cases} u_s = \psi_{++} \quad \text{others zero} \\ u_s = \psi_{+-} \quad \text{"} \quad \text{"} \\ u_s = \psi_{-+} \quad \text{"} \quad \text{"} \\ u_s = \psi_{--} \quad \text{"} \quad \text{"} \end{cases}$$

$$\boxed{\begin{matrix} \frac{1}{\sqrt{2}} u_s = \psi_{+-} \\ \frac{1}{\sqrt{2}} u_s = \psi_{-+} \end{matrix}}$$

Helium spectrum



Exp	Theor
198,308	198,298 ± 8 cm^{-1}

$\boxed{\text{Ferromagnetism}}$

$\boxed{\text{Para-ortho hydrogen}}$

~~Ortho & Para hydrogen~~ Periodic System

H

He

Li Be B. C N. O F. Ne

Na Mg Al Si P. S Cl .A

K Ca Sc Ti V Cr Mn Fe Co Ni

Cu Zn Ga Ge As Se Br Kr

Rb Sr Y Zr Nb Mo (43) Ru Rh Pd

Ag Cd In Sn Sb Te I Xe

Cs Ba [La Ce Pr Nd Sm Eu Gd Tb Dy Ho Er Tu Yb Lu]

Hf Ta W Re Os Ir Pt

Au Hg Tl Pb Bi Po (85) Em

(87) Ra Ac Th Pa U

54		1s	2s	2p	3s	3p	3d	4s	4p	4d	4f	5s	5p	5d	6s	6p
2	He	2														
10	Ne	2	2	6												
18	A	2	2	6	2	6										
36	Kr	2	2	6	2	6	10	2	6							
54	X	2	2	6	2	6	10	2	6	10		2	6			
86	Em	2	2	6	2	6	10	2	6	10	14	2	6	10	2	6

Para ortho hydrogen

Atom in the magnetic field.

Perturbations with quasi degeneracy

$w_1 \cong w_2 \qquad w_3$

$u_1^{(0)} \quad u_2^{(0)} \quad u_3^{(0)} \quad u_4^{(0)}$

$$u = c_1 u_1^{(0)} + c_2 u_2^{(0)} + c_3 u_3^{(0)} + c_4 u_4^{(0)}$$

$c_1\ ,\ c_2$ order of magnitude 1

$c_3 \ll 1 \qquad c_4 \ll 1 \ldots$

$$(H_0 + \mathcal{H})\left(c_1 u_1^{(0)} + c_2 u_2^{(0)} + c_3 u_3^{(0)} + \ldots\right) = w c_1 u_1^{(0)} + w c_2 u_2^{(0)} + w c_3 u_3^{(0)} + \ldots$$

$$c_1 w_1^{(0)} u_1^{(0)} + c_2 w_2^{(0)} u_2^{(0)} + c_3 w_3^{(0)} u_3^{(0)} + \ldots + c_1 \mathcal{H} u_1^{(0)} + c_2 \mathcal{H} u_2^{(0)} + c_3 \mathcal{H} u_3^{(0)} + \ldots = w c_1 u_1^{(0)} + w c_2 u_2^{(0)} + w c_3 u_3^{(0)} + \ldots$$

$$c_1 (w_1^{(0)} - w) u_1^{(0)} + c_2 (w_2^{(0)} - w) u_2^{(0)} + c_3 (w_3^{(0)} - w) u_3^{(0)} + + c_1 \mathcal{H} u_1^{(0)} + c_2 \mathcal{H} u_2^{(0)} + \text{higher terms} = 0$$

Exam ~~W~~ Winter 1937

An electron is bound to move on a spherical surface of radius 10^{-8} cm. Find its energy levels and the wavelength of the radiation emitted in the transition between the two lowest energy levels

Discuss the ~~ff~~ Schrödinger equation of a mass point that moves on a plane in a central field of force. ~~(discussion analogous to that~~ ~~for a~~

A linear harmonic oscillator is perturbed by a force with potential energy κx^4. Find the first order perturbation of the first and second energy levels.

An electron movable on a spherical surface is perturbed by a constant electric field. Find the perturbation of the second ~~f~~ energy level.

The four operators

$$x \; , \; \frac{d}{dx} \; , \; \frac{d^2}{dx^2} \; , \; \log x$$

act on functions of x.

~~State what~~ For all pairs ~~any~~ two of them state whether or not they commute. If they dont commute find the difference between the products taken in different order

Pauli's theory of spin

$\sigma_x \quad \sigma_y \quad \sigma_z$ are operators on the [illegible] functions of two valued spin variable with eigenvalues ± 1

$$\sigma_z u_+ = u_+ \qquad \text{angular momentum} = \frac{h}{4\pi}\sigma$$

$$\sigma_z u_- = -u_- \qquad \sigma \times \sigma = 2i\sigma$$

$$\sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

$$\sigma_x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \qquad \sigma_y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}$$

Example with α, β, γ cosines of direction

~~Measurement of~~ Eigenvalues of

$$\sigma_n = \alpha\sigma_x + \beta\sigma_y + \gamma\sigma_z = \begin{vmatrix} \gamma & \alpha - i\beta \\ \alpha + i\beta & -\gamma \end{vmatrix}$$

are ± 1

If measurement gives $\sigma_n = +1$

Then

$$\gamma\psi_+ + (\alpha - i\beta)\psi_- = \psi_+$$

$$(\alpha + i\beta)\psi_+ - \gamma\psi_- = \psi_-$$

$$\begin{vmatrix} \psi_+ \\ \psi_- \end{vmatrix} \text{ proportional to } \begin{vmatrix} 1+\gamma \\ \alpha + i\beta \end{vmatrix} \qquad \text{normalized} \quad \frac{1}{\sqrt{2(1+\gamma)}} \begin{vmatrix} 1+\gamma \\ \alpha + i\beta \end{vmatrix}$$

$1+\gamma^2+2\gamma \qquad 2(1+\gamma)$

$$\frac{(1+\gamma)^2}{2(1+\gamma)} = \frac{1+\cos\vartheta}{2} = \cos^2\frac{\vartheta}{2}$$

$$\frac{\alpha^2+\beta^2}{2(1+\gamma)} = \frac{1-\cos\vartheta}{2} = \sin^2\frac{\vartheta}{2}$$

Hamiltonian of electron in e.m. field
(non relativistic)

$$\frac{1}{2m}\left(p-\frac{e}{c}A\right)^2 + eV = W$$

$$\dot{x} = \frac{\partial W}{\partial p_x} = \frac{1}{m}\left(p_x - \frac{e}{c}A_x\right) \qquad m\dot{x} + \frac{e}{c}A_x = p_x$$

$$\dot{p}_x = m\ddot{x} + \frac{e}{c}\left(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right) =$$

$$= -\frac{\partial W}{\partial x} = -e\frac{\partial V}{\partial x} + \frac{e}{c}\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right)$$

$$m\ddot{x} = eE + \frac{e}{c}\left(\dot{y}H_z - \dot{z}H_y\right)$$

Uniform magnetic field $H \parallel$ to z

$$A_x = -\frac{H}{2}x \qquad W = \frac{p^2}{2m} + eV + \frac{eH}{2mc}\left(yp_x - xp_y\right)$$

$$A_y = +\frac{H}{2}y$$

$$\frac{p^2}{2m} + eV + \frac{eH}{2mc}(yp_x - xp_y)$$

Normal Zeeman effect

Bohr Magneton

Apparent field due to spin orbit interaction

$$\frac{Ze}{r^2}\frac{v}{c}\sin\vartheta = \frac{Ze}{mc}\frac{m\,\bar{v}\times\bar{r}}{r^3}$$

$$= \frac{Ze}{mc}\frac{\vec{M}}{r^3}$$

Thomas factor $\frac{1}{2}$

In general for electron in central field

Apparent field

$$\left(\frac{1}{2}\right)\frac{v}{c}E(r)\sin\vartheta = \frac{1}{mc}\frac{E(r)}{r}M\times\left(\frac{1}{2}\right)$$

Complete Hamilton function for electron in central field V & uniform magnetic field ∥ z

$$\frac{p^2}{2m} + eV(r) - \frac{eH}{2mc}M_z - \frac{ehH}{4\pi mc}\sigma_z +$$

$$+ \frac{1}{2mc}\frac{1}{r}\frac{\partial V}{\partial r}\frac{eh}{4\pi mc}(M\sigma)$$

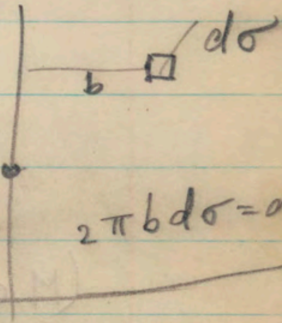
Bohr magneton from current density

(14) (15)

$$J = \frac{eh}{4\pi m i c}\left(\psi^* \operatorname{grad}\psi - \psi \operatorname{grad}\psi^*\right)$$

$$\psi = e^{im\varphi}\, v(r,\vartheta)$$

$$J = \frac{eh}{4\pi m i c}\, v^2(r,\vartheta)\, \frac{2im}{b}$$



$$\mu = \int J\, d\sigma \times \pi b^2 = \frac{eh}{4m\pi c}\, m \int v^2\, d\tau \; ; \; \frac{eh}{4\pi m c}\, \underline{\underline{m}}$$

half merid plane

$$2\pi b\, d\sigma = d\tau$$

Spin in magnetic field

$$\frac{eh}{4\pi m c}\left(H_x\sigma_x + H_y\sigma_y + H_z\sigma_z\right)$$

Description of the spin motion in a magnetic field

$$\begin{vmatrix} \psi_+ \\ \psi_- \end{vmatrix} \qquad \begin{vmatrix} \gamma & \alpha - i\beta \\ \alpha + i\beta & -\gamma \end{vmatrix} \begin{vmatrix} \psi_+ \\ \psi_- \end{vmatrix} = \begin{vmatrix} \psi_+ \\ \psi_- \end{vmatrix}$$

$$\gamma\psi_+ + (\alpha - i\beta)\psi_- = \psi_+$$

$$(\alpha + i\beta)\psi_+ - \gamma\psi_- = \psi_-$$

Discussion of

$$\frac{p^2}{2m} + eV(r) + \overbrace{\frac{\mu_0}{2mc}\frac{1}{r}\frac{\partial V}{\partial r}}^{f(r)}(M\cdot\sigma) = \mathcal{H}$$

$$\frac{\hbar}{i}\dot{M} = \frac{i}{\hbar}\left(\mathcal{H}M - M\mathcal{H}\right) = \cancel{\frac{\mu_0}{2mc}}\ \cancel{\frac{1}{r}} \ f(r)(M\cdot\sigma)M - M f(r)(M\cdot\sigma)$$

$$\cancel{M_x\sigma} \quad f(r)M_x - M_x f(r)$$

$$f(r)(y p_z - z p_y) - (y p_z - z p_y) f(r) = 0$$

$$f(r)p_z - p_z f(r) = -\frac{\hbar}{i} f'(r)\frac{z}{r}$$

$$(M\cdot\sigma)M - M(M\sigma)$$

$$(M_x\sigma_x + M_y\sigma_y + M_z\sigma_z)M_x - M_x(\qquad\qquad)$$

$$\sigma\times\sigma = 2i\sigma \qquad M\times M = -\frac{\hbar}{i}M$$

$$+\frac{\hbar}{i}\left(M_z\sigma_y - M_y\sigma_z\right) \qquad (M\cdot\sigma)M - M(M\cdot\sigma) =$$

$$= -\frac{\hbar}{i}\{M\times\sigma\}$$

$\lfloor r$

Conservation of angular momentum

$$(M\sigma)\sigma - \sigma(M\sigma) = 2i\sigma$$

Motion of orientation of spin in magnetic field

$$\psi_+ \sim 1+\gamma = 1+\cos\vartheta$$

$$\psi_- \sim \alpha + i\beta = \sin\vartheta \, e^{i\varphi} \qquad \frac{\psi_+}{\psi_-} = \frac{\sin\vartheta}{1-\cos\vartheta} e^{-i\varphi}$$

$$\cancel{\frac{\psi_-}{\psi_+} = \frac{\sin\vartheta}{1+\cos\vartheta} e^{i\varphi}} \qquad \frac{\psi_-}{\psi_+} = \frac{\sin\vartheta}{1+\cos\vartheta} e^{i\varphi}$$

$$\cancel{\psi_+ = \frac{1}{\sqrt{2(1+\cos\vartheta)}}} \qquad \frac{\cancel{(1+\gamma)}(\alpha - i\beta)}{1-\gamma}$$

$$\begin{vmatrix} H_z & H_x - iH_y \\ H_x + iH_y & -H_z \end{vmatrix} \begin{vmatrix} \psi_+ \\ \psi_- \end{vmatrix} = -\frac{\hbar}{i\mu} \begin{vmatrix} \dot\psi_+ \\ \dot\psi_- \end{vmatrix}$$

$$\dot\psi_+ = -\frac{i\mu}{\hbar}\left(H_z\psi_+ + (H_x - iH_y)\psi_-\right)$$

$$\dot\psi_- = -\frac{i\mu}{\hbar}\left((H_x + iH_y)\psi_+ - H_z\psi_-\right)$$

$$\frac{d}{dt}\frac{\psi_-}{\psi_+} = \frac{\dot\psi_-}{\psi_+} - \frac{\psi_-\dot\psi_+}{\psi_+^2} = -\frac{i\mu}{\hbar}\left[(H_x + iH_y) - 2H_z\frac{\psi_-}{\psi_+} - \right.$$

$$\left. - (H_x - iH_y)\frac{\psi_-^2}{\psi_+^2}\right]$$

$$l_x = \frac{1}{\hbar} M_x \qquad l_y = \frac{1}{\hbar} M_y \qquad l_z = \frac{1}{\hbar} M_z$$

$$\vec{l} \times \vec{l} = i\,l \qquad l_x = \frac{1}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$l_z\, Y_{lm}(\vartheta,\varphi) = m\, Y_{lm}(\vartheta,\varphi)$$

$$(l_x + i\,l_y)\, Y_{lm}(\vartheta,\varphi) = -\sqrt{(l-m)(l+m+1)}\; Y_{l,m+1}(\vartheta,\varphi)$$

$$(l_x - i\,l_y)\, Y_{lm}(\vartheta,\varphi) = -\sqrt{(l+m)(l-m+1)}\; Y_{l,m-1}(\vartheta,\varphi)$$

$$\cancel{l^2} \quad \vec{l} + \tfrac{1}{2}\vec{\sigma} = j$$

$$l_z + \tfrac{1}{2}\sigma_z = m$$

$$\vec{l}^{\,2},\ \vec{j}^{\,2},\ m,\ \cancel{(l,\sigma)}$$

commute among each other and with the energy operator

$$\mathcal{H} = \frac{p^2}{2m} - eV + f(r)\,(l\sigma)$$

~~take therefore~~ $$f(r) = -\frac{\hbar\mu_0}{2mc}\,\frac{1}{r}\,\frac{\partial V}{\partial r}$$

$$\cancel{\vec{l}^{\,2} = l(l+1)} \qquad \cancel{m = m} \text{ ~~Mes~~}$$

$$\cancel{\vec{\sigma}^{\,2} = 3}$$

$$\cancel{\vec{j}^{\,2} = l(l+1) + \frac{3}{4} + (l\cdot\sigma)}$$

Eigenvalues of ϑ_m are ~~half integral half~~ integral $+\frac{1}{2}$ numbers (half odd numbers)
Hence to l, m corresponds e.f.

$$\begin{vmatrix} R\, Y_{l, m-\frac{1}{2}} \\ Q\, Y_{l, m+\frac{1}{2}} \end{vmatrix}$$

(l, σ) has eigenvalues $\begin{matrix} l \\ -(l+1) \end{matrix}$

Corresponding e.f. are (normalised)

$$(l, l+\tfrac{1}{2}, m) = \frac{1}{\sqrt{2l+1}} \begin{vmatrix} \sqrt{l+\frac{1}{2}+m}\; Y_{l, m-\frac{1}{2}} \\ -\sqrt{l+\frac{1}{2}-m}\; Y_{l, m+\frac{1}{2}} \end{vmatrix} \qquad \text{for } (l\sigma) = l \quad j = l+\tfrac{1}{2}$$

and

$$(l, l-\tfrac{1}{2}, m) = \frac{1}{\sqrt{2l+1}} \begin{vmatrix} \sqrt{l+\frac{1}{2}-m}\; Y_{l, m-\frac{1}{2}} \\ \sqrt{l+\frac{1}{2}+m}\; Y_{l, m+\frac{1}{2}} \end{vmatrix} \qquad \text{for } (l\sigma) = -l-1 \quad j = l-\tfrac{1}{2}$$

generalised spin functions (note special case of $l=0$

effective potential

$$-eV + l f(r) \qquad \text{for } j = l+\tfrac{1}{2}$$

$$-eV - (l+1) f(r) \qquad \text{for } j = l-\tfrac{1}{2}$$

Sistematic spectroscopic notations

A is an operator
Classify the functions (vectors of Hilbert space) calling a class of functions (or vectors) those that fulfill

$$A\psi = a\psi \qquad (\text{Class } A=a)$$

If ~~a~~ a is a non-degenerate e.v. of A there ~~is~~ the class is made by the vectors parallel to one – When there is degeneracy the class is a subspace of the Hilbert space of ~~one~~ more than one dimension

Theorem:
~~If~~

$$AB = BA$$

and ψ belongs to the $A=a$ class also

$$B\psi$$

belongs to the $A=a$ class

Proof

$$A\psi = a\psi$$

$$BA\psi = aB\psi$$

$$A(B\psi) = a(B\psi) \qquad \text{Q.E.D.}$$

Since $B\psi$ ~~is a linear combination~~ Belongs to the $A=a$ class we may consider the operator B in this Hilbert sub-space

There B will have eigen values and e.f. in number equal to the dimensions of the sub-space - They are obtained as solutions of the system

$$\begin{cases} A\psi = a\psi \\ B\psi = b\psi \end{cases}$$

Such a system defines a sub-class of the class $A=a$ that may have any number of dimensions ≥ 1 and $\leq$ the no of dimensions of class $A=a$; we call this sub class class $A=a, B=b$

Let C be a third operator that commutes with both A and B

$$AC=CA \quad , \quad BC=CA$$

All we said of B with respect to class $A=a$ holds of C with respect to the class $A=a, B=b$

And so forth

Example $\quad \mathcal{H}_0 = \frac{p^2}{2m} - eV(r) + f(r)(l,\sigma)$

~~group of~~ The operators $\mathcal{H}_0$ ~~l_z~~ $\vec{l}^2$ ~~l_z~~ $\vec{j}^2$ $j_z = m$ intercommute (hence classification of terms)

From spin spherical functions follows

$$\overline{\sigma_z} = \pm \frac{2m}{2l+1} \quad \text{for } j = l \pm \tfrac{1}{2}$$

Zeeman effect
perturbation

$$W_{mag} = H\mu_0 (l_z + \sigma_z) = H\mu_0 \left(m + \tfrac{1}{2}\sigma_z\right)$$

Since m commutes with the perturbation we can neglect the degeneracy; hence in first approximaton

$$\overline{W}_{mag} = H\mu_0 m \left(1 \pm \frac{1}{2l+1}\right) \quad \text{for } j = l \pm \tfrac{1}{2}$$

Discussion: Semiclassical derivation of g factors of Landé

$$g = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}$$

$$\sigma_z\left(l, l+\tfrac{1}{2}, m\right) = \frac{2m}{2l+1}\left(l, l+\tfrac{1}{2}, m\right) + \frac{2\sqrt{(l+\frac{1}{2})^2 - m^2}}{2l+1}\left(l, l-\tfrac{1}{2}, m\right)$$

$$\sigma_z\left(l, l-\tfrac{1}{2}, m\right) = \frac{2\sqrt{(l+\frac{1}{2})^2 - m^2}}{2l+1}\left(l, l+\tfrac{1}{2}, m\right) - \frac{2m}{2l+1}\left(l, l-\tfrac{1}{2}, m\right)$$

Perturbation matrix of Paschen-Back effect is

$$H\mu_0\left(m + \tfrac{1}{2}\sigma\right) + \mathcal{H}_0$$

$$H\mu_0 m + \begin{vmatrix} w & 0 \\ 0 & 0 \end{vmatrix} + \frac{H\mu_0}{2l+1}\begin{vmatrix} m & \sqrt{(l+\frac{1}{2})^2 - m^2} \\ \sqrt{} & -m \end{vmatrix}$$

w = energy difference between terms with $j = l+\frac{1}{2}$ and $j = l-\frac{1}{2}$

Put

$$a = \frac{2l+1}{H\mu_0}w$$

$$\begin{vmatrix} m+a & \sqrt{} \\ \sqrt{} & -m \end{vmatrix} \text{ has e.values } \frac{a \pm \sqrt{a^2 + 4am + 4(l+\frac{1}{2})^2}}{2}$$

70

Identical particles

Operator H is symmetrical

$H\psi_{symm}$ is symm

$H\psi_{antis}$ is antisymm

Particles with symm and antis wave functions.

Individual quantum states

$$H = H_1 + H_2$$

---

a) System of independent particles

$SH = HS$

Symmetrical operators

S = symm operator

---

a Independent particles
Energy sum
Wave functions products
Individual quantum states

b Symmetry properties of operators
Symmetry operation S
$SA = AS$ is equivalent to say that A has the symmetry S
Examples (rotation of space — Reflection
exchange of identical particles

See pages 50 – 54

$H = H_1 + H_2 + \mathcal{H}_{12}$

$H_1(q_1)$ $\quad H_2 = H_1(q_1)$ $\quad \mathcal{H}_{12}$ simmetrical

Degeneracy with individual states

$\psi_n(x_1)\,\psi_m(x_2) \qquad \psi_m(x_1)\,\psi_n(x_2) \qquad E = W_n + W_m$

Several particles

Definition of symm and antis

$H\psi = W\psi$

Case of individual quantum states

$\sum_{(P)} \psi_{n_{P1}}(x_1)\,\psi_{n_{P2}}(x_2)\cdots\psi_{n_{PA}}(x_A) \qquad \left[(-1)^P\right]$

Pauli principle (go back pages 50-54)

statistics (50-51)

Pauli principle with Spin (p. 51)

Para & orthohelium (p. 52, 53)

Periodic system p 53 – 54

Diracs Wave function

$$\gamma_x = \begin{vmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{vmatrix} \qquad \gamma_y = \begin{vmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{vmatrix}$$

$$\gamma_z = \begin{vmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{vmatrix} \qquad \delta = \begin{vmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{vmatrix}$$

$$W = eV - c\left(\gamma \cdot p - \frac{eA}{c}\right) - mc^2\delta$$

$$(W - eV)^2 = c^2\left(p - \frac{eA}{c}\right)^2 + m^2c^4$$

$W\psi_1$ $$-\frac{\hbar}{i}\frac{\partial\psi_1}{\partial t} = \overset{(eV-mc^2)\psi_1}{\cancel{eV\psi_1}} - \frac{c\hbar}{i}\left[\frac{\partial\psi_4}{\partial x} - i\frac{\partial\psi_4}{\partial y} + \frac{\partial\psi_3}{\partial z}\right] + e\left[(A_x - iA_y)\psi_4 + A_z\psi_3\right]$$

$W\psi_2$ $$-\frac{\hbar}{i}\frac{\partial\psi_2}{\partial t} = \overset{(eV-mc^2)\psi_2}{\cancel{eV\psi_2}} - \frac{c\hbar}{i}\left[\frac{\partial\psi_3}{\partial x} + i\frac{\partial\psi_3}{\partial y} - \frac{\partial\psi_4}{\partial z}\right] + e\left[(A_x + iA_y)\psi_3 - A_z\psi_4\right]$$

$W\psi_3$ $$-\frac{\hbar}{i}\frac{\partial\psi_3}{\partial t} = \overset{(eV+mc^2)\psi_3}{\cancel{eV\psi_3}} - \frac{c\hbar}{i}\left[\frac{\partial\psi_2}{\partial x} - i\frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_1}{\partial z}\right] + e\left[(A_x - iA_y)\psi_2 + A_z\psi_1\right]$$

$W\psi_4$ $$-\frac{\hbar}{i}\frac{\partial\psi_4}{\partial t} = \overset{(eV+mc^2)\psi_4}{\cancel{eV\psi_4}} - \frac{c\hbar}{i}\left[\frac{\partial\psi_1}{\partial x} + i\frac{\partial\psi_1}{\partial y} - \frac{\partial\psi_2}{\partial z}\right] + e\left[(A_x + iA_y)\psi_1 - A_z\psi_2\right]$$

Plane waves

W near mc^2 $\qquad$ $W = mc^2 + E$

$$E \ll mc^2$$

$$\psi_1 = -\frac{e\hbar}{2mic}\left[\frac{\partial\psi_4}{\partial x} - i\frac{\partial\psi_4}{\partial y} + \frac{\partial\psi_3}{\partial z}\right]$$

$$\psi_2 = -\frac{\hbar}{2imc}\left[\frac{\partial\psi_3}{\partial x} + i\frac{\partial\psi_3}{\partial y} - \frac{\partial\psi_4}{\partial z}\right]$$

~~Plane waves~~

$$E\psi_3 = eV\psi_3 + \frac{c\hbar}{i}\frac{\hbar}{2mic}\Delta\psi_3$$

$$2mc^2\left(1+\frac{E-eV}{2mc^2}\right)$$

$$\left(2mc^2 + E - eV\right)\psi_1 = -\frac{c\hbar}{i}\left[\frac{\partial\psi_4}{\partial x} - \dots\right] + e\left[(A_x - iA_y)\psi_4 + \dots\right]$$

$$\psi_1 = -\frac{\hbar}{2imc}\left[\frac{\partial\psi_4}{\partial x} - \dots\right] + \frac{e}{2mc^2}\left[(A_x - iA_y)\psi_4 + A_z\psi_3\right]$$

~~Check~~

$$(E - eV)\psi_3 = -\frac{\hbar^2}{2m}\Delta\psi_3 - \frac{e\hbar}{2mci}\left[(A_x - iA_y)\frac{\partial\psi_3}{\partial x} + \dots\dots\right]$$

$$\frac{\partial \psi_3}{\partial x} + i \frac{\partial \psi_3}{\partial y} - \frac{\partial \psi_4}{\partial z} \qquad (A_x + i A_y)\psi_3 - A_z \psi_4$$

ψ_4

$$2mc^2\left(1 + \frac{E-eV}{2mc^2}\right)\psi_1 = -\frac{c\hbar}{i}\left(\frac{\partial \psi_4}{\partial x} - i\frac{\partial \psi_4}{\partial y} + \frac{\partial \psi_3}{\partial z}\right) + e\left[(A_x - i A_y)\psi_4 + A_z \psi_3\right]$$

$$\psi_1 = -\frac{\hbar}{2imc}\left(\frac{\partial \psi_4}{\partial x}\cdots\right) + \frac{e}{2mc^2}\left(A_x \psi_4 \cdots\right) + \frac{E-eV}{2mc^2}\frac{\hbar}{2imc}\left(\frac{\partial \psi_4}{\partial x} + \ldots\right)$$

$$\psi_2 = -\frac{\hbar}{2imc}\left(\frac{\partial \psi_3}{\partial x}\cdots\right) + \frac{e}{2mc^2}\left(A_x \psi_3 \cdots\right) + \frac{E-eV}{2mc^2}\frac{\hbar}{2imc}\left(\frac{\partial \psi_3}{\partial x} + \ldots\right.$$

$$(E-eV)\psi_3 = -\frac{c\hbar}{i}\left(\frac{\partial \psi_2}{\partial x} - i\frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_1}{\partial z}\right) + e\left[(A_x - i A_y)\psi_2 + A_z \psi_1\right]$$

$$(E-eV)\psi_3 = -\frac{\hbar^2}{2m}\Delta\psi_3 - \frac{e\hbar}{2mci}\left[A_x \frac{\partial \psi_3}{\partial x} + \frac{\partial A_x}{\partial x}\psi_3 + \right.$$

$$+ i A_y \frac{\partial \psi_3}{\partial x} + i\frac{\partial A_y}{\partial x}\psi_3 - A_z\frac{\partial \psi_4}{\partial x} - \frac{\partial A_z}{\partial x}\psi_4 - i A_x \frac{\partial \psi_3}{\partial y} - i\frac{\partial A_x}{\partial y}\psi_3$$

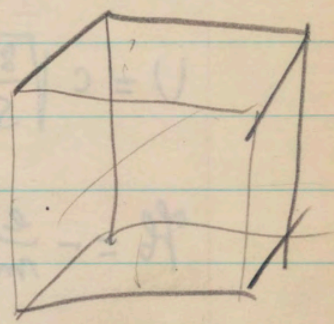
$$+ A_y \frac{\partial \psi_3}{\partial y} + \frac{\partial A_y}{\partial y}\psi_3 + i\frac{\partial A_z}{\partial y}\psi_4 - i A_z \frac{\partial \psi_4}{\partial y} + A_x \frac{\partial \psi_4}{\partial z} + \frac{\partial A_x}{\partial z}\psi_4$$

$$- i A_y \frac{\partial \psi_4}{\partial z} - i\frac{\partial A_y}{\partial z}\psi_4$$

$$dN = \frac{8\pi}{c^3}\,\Omega\,\nu^2 d\nu$$

$$U = A\,u(t)\sin\left(\frac{2\pi\nu}{c}(\alpha\cdot r)+\beta\right)$$

$$U = \sum_s A_s u_s(t)\sin\Gamma_s = \sum U_s$$



$$\Gamma_s = \frac{2\pi\nu_s}{c}\,\alpha_s\cdot r + \beta_s$$

$$E = -\frac{1}{c}\frac{\partial U}{\partial t} \qquad H = \mathrm{rot}\,U$$

$$E = -\frac{1}{c}\sum_s A_s \dot{u}_s \sin\Gamma_s$$

$$H = \sum_s \frac{2\pi\nu_s}{c}[\alpha_s\times A_s]\,u_s\cos\Gamma_s$$

$$\overline{E^2} = \frac{1}{2c^2}\sum_s \dot{u}_s^2 \qquad \overline{H_s^2} = \sum \frac{2\pi^2\nu_s^2}{c^2}u_s^2$$

$$W_e = \frac{\Omega}{8\pi c^2}\sum_s$$

$$\nu = \frac{c}{a}\left(n_x^2+n_y^2+n_z^2\right)$$

$$\cancel{\frac{2\pi}{a}(n_x u_x + \dots) = \frac{2\pi\nu}{c}}$$

$$\cancel{\frac{\pi}{6}\frac{a^3\nu^3}{c^3} \quad \frac{\pi}{2}\frac{a^3}{c^3}\nu^2 d\nu \quad \frac{2\pi}{8}\left(\frac{a}{c}\nu\right)^2\frac{a}{c}d\nu}$$

$$8\pi\frac{a^3}{c^3}\nu^2 d\nu$$

$$H = \frac{p^2}{2m} + eV + \sum_s \left(\frac{p_s^2}{2} + 2\pi^2 \nu_s^2 q_s^2 \right) + \mathcal{H}$$

$$\mathcal{H} = -\frac{e}{mc} (U \cdot p)$$

$$U = c \sqrt{\frac{8\pi}{\Omega}} \sum_s A_s \, q_s \sin \Gamma_s$$

$$\mathcal{H} = -\frac{e}{m} \sqrt{\frac{8\pi}{\Omega}} \sum_s (A_s \cdot p) \, q_s \sin \Gamma_s$$

$$\int q_s u_{n_s} u_{m_s} \, dq_s = \begin{cases} 0 & \text{for } m_s \neq n_s \pm 1 \\ \sqrt{\frac{h}{8\pi^2 \nu_s}} \sqrt{n_s + 1} & m_s = n_s + 1 \\ \sqrt{\frac{h}{8\pi^2 \nu_s}} \sqrt{n_s} & m_s = n_s - 1 \end{cases}$$

$$\mathcal{H}_{n \, n_s ; m \, n_s \pm 1} = -\frac{e}{m} \sqrt{\frac{8\pi}{\Omega}} \sqrt{\frac{h}{8\pi^2 \nu_s}} \left| \begin{matrix} \sqrt{n_s + 1} \\ \text{or} \\ \sqrt{n_s} \end{matrix} \right| (A_s \cdot P_{snm})$$

$$P_{snm} = \int u_n^* (\sin \Gamma_s p) \, u_m \, d\tau =$$

$$= \frac{h}{2\pi i} \int \sin \Gamma_s \, u_n^* \frac{\partial u_m}{\partial r} \, d\tau$$

$$\int u_m^* \frac{\partial u_n}{\partial x}\, d\tau =$$

$$\Delta u_n + \frac{8\pi^2 m}{h^2}\left(E_n - eV\right) u_n = 0$$

$$\Delta u_m^* + \frac{8\pi^2 m}{h^2}\left(E_m - eV\right) u_m^* = 0$$

$$u_m^* \Delta u_n - u_n \Delta u_m^* = \frac{8\pi^2 m}{h^2}\left(E_n - E_m\right) u_n u_m^*$$

$$\operatorname{div}\left(u_m^* \operatorname{grad} u_n - u_n \operatorname{grad} u_m^*\right) = \frac{8\pi^2 m}{h^2}\left(E_n - E_m\right) u_n u_m^*$$

$$x \operatorname{div}\left(\qquad\right) = \frac{8\pi^2 m}{h^2}\left(E_n - E_m\right) x u_n u_m^*$$

$$-\int\left(u_m^* \frac{\partial u_n}{\partial x} - u_n \frac{\partial u_m^*}{\partial x}\right) d\tau = \frac{8\pi^2 m}{h^2}\left(E_n - E_m\right) x_{mn}$$

$$-\int u_m^* \frac{\partial u_n}{\partial x}\, d\tau = \frac{4\pi^2 m}{h^2}\left(E_n - E_m\right) x_{mn}$$

$$-\frac{2\pi i}{h} p_{mn} = \frac{4\pi^2 m}{h^2}\left(E_n - E_m\right) x_{mn}$$

$$p_{mn} = 2\pi i m \nu_{nm} x_{mn}$$

$$\dot a_{n n_s} = \frac{4\pi^{3/2} e}{\sqrt{h\Omega}} \sum_{ms} \frac{\nu_{mn}}{\sqrt{\nu_s}} \left(A_s\cdot r_{nm}\right) \sin\Gamma_s \times$$

$$\times\left[a_{m\, n_s+1}\sqrt{n_s+1}\, e^{-2\pi i(\nu_{mn}+\nu_s)t} + a_{m\, n_{s-1}}\sqrt{n_s}\, e^{-2\pi i(\nu_{mn}-\nu_s)t}\right]$$

Emission of radiation

$$a_{m0}(0) = 1$$

$$\dot a_{n1_s} = \frac{4\pi^{3/2} e}{\sqrt{h\Omega}} \frac{\nu_{mn}}{\sqrt{\nu_s}} \left(A_s\cdot r_{nm}\right) \sin\Gamma_s\, e^{-2\pi i(\nu_{mn}-\nu_s)t}$$

$$a_{n1_s} = \frac{4\pi^{3/2} e}{\sqrt{h\Omega}} \frac{\nu_{mn}}{\sqrt{\nu_s}} \left(A_s\cdot r_{nm}\right) \sin\Gamma_s \frac{e^{-2\pi i(\nu_{mn}-\nu_s)t}-1}{-2\pi i(\nu_{mn}-\nu_s)}$$

$$\left|a_{n1_s}\right|^2 = \frac{16\pi e^2}{h\Omega} \frac{\nu_{mn}^2}{\nu_s} \left|A_s\cdot r_{nm}\right|^2 \sin^2\Gamma_s \frac{\sin^2 \pi t(\nu_{mn}-\nu_s)}{(\nu_{mn}-\nu_s)^2}$$

H Mittleman

Physics 238 Dec ~~Sept~~ 1940

1) Discuss the meaning of the following Pauli operators:

$$\sigma_z , \quad \sigma_x + i\,\sigma_y , \quad \sigma_x \sigma_y \sigma_z .$$

2) Discuss the electronic configurations of the fundamental states of the following elements: Al $(Z=13)$; Kr $(Z=36)$.

3) Discuss the lowest energy levels of the element Ca $(Z=20)$.

4) Find the Zeeman effect pattern of a $^2F_{5/2}$ and $^2F_{7/2}$ terms of the spectrum of an alkali atom.

Born theory
Born + classical theory for Coulomb field
Exact collision theory
Born theory of anelastic collision
Born theory of collisions in which a particle is absorbed and a different particle is emitted (n, γ) - processes
Compound nucleus

Physics 247 (Winter 1940-41) 81

Collision theory

$$\dot{a}_{k} = -\frac{i}{\hbar}\sum_{m} \mathcal{H}_{kj}\, a_{j}\, e^{\frac{i}{\hbar}(W_k - W_j)t}$$

prob of transition $= \frac{2\pi}{\hbar}\left|\mathcal{H}_{kj}\right|^2 n$ ~~number of states per unit energy~~

Born approximation

$$\psi_{j} = \frac{1}{\sqrt{\Omega}} e^{\frac{i}{\hbar} p\cdot r}$$

$$\psi_{k} = \frac{1}{\sqrt{\Omega}} e^{\frac{i}{\hbar} q\cdot r}$$

$$\mathcal{H}_{kj} = \frac{1}{\Omega}\int V(r)\, e^{\frac{i}{\hbar}(\vec{p}-\vec{q}\cdot\vec{r})}$$

no of states $\frac{\Omega}{h^3} p^2\, dp\, d\omega$

energy interval $v\, dp$

$$n = \frac{\Omega}{8\pi^3\hbar^3}\frac{p^2}{v} d\omega$$

prob of transition $= \frac{1}{\Omega} v\, \sigma_{d\omega} = \frac{2\pi}{\hbar}\frac{1}{\Omega^2}\left|\int V e^{\frac{i}{\hbar}(\vec{p}-\vec{q}\cdot r)} d\tau\right|^2 \times \frac{\Omega}{8\pi^3\hbar^3}\frac{p^2}{v} d\omega$

$$\sigma_{d\omega} = \frac{1}{4\pi^2\hbar^4}\frac{p^2}{v^2}\left|\int V e^{\frac{ip}{\hbar}(\vec{j}-\vec{k}\cdot\vec{r})} d\tau\right|^2 d\omega$$

82

Conditions:

no substantial perturbation of the e.f. hence either

a) very large energy

$$W \gg |V|$$

b) for small energy: small radius of action ρ and precisely

$$\frac{\hbar}{\sqrt{mV}} \gg \rho \quad \text{or} \quad \frac{mV\rho^2}{\hbar^2} \ll 1$$

Condition b) is equivalent to

$$V^2 \rho^6 \frac{m^2}{\hbar^4} < \rho^2$$

scattering cross-section ≪ geometrical cross-section

Special cases

Coulomb field

$$V = \frac{Ze^2}{r}$$

$$Ze^2 \int \frac{1}{r} e^{\frac{i}{\hbar} P.r} = \frac{4\pi\hbar^2}{P^2} Ze^2 \qquad P = 2p \sin\frac{\theta}{2}$$

$$-\frac{P^2}{\hbar^2} \cdots = + \frac{4\pi Ze^2 \hbar^2}{P^2}$$

$$\sigma_{d\omega} = \frac{m^2 d\omega}{4\pi^2\hbar^4} \frac{16\pi^2\hbar^4 Z^2 e^4}{16 p^4 \sin^4\frac{\theta}{2}} =$$

Limit to validity

$$\frac{Ze^2}{mv^2} \ll \frac{\hbar}{mv} \quad \text{or} \quad \frac{Ze^2}{\hbar v} \ll 1$$

Limit of classical theory

$$\frac{Ze^2}{r^2}\frac{r}{v} \gg \frac{\hbar}{\cancel{mv}\, r} \quad \text{or} \quad \frac{Ze^2}{\hbar v} \gg 1$$

hence two limitations are complementary since result is the same for Born & classical theory it may be expected to be exact

---

Small energy: isotropic scattering

$$\sigma = \frac{m^2}{\pi \hbar^4}\left|\int V d\tau\right|^2$$

valid provided $\sigma \ll \rho^2$

Large energies: scattering within angle

$$\frac{p}{\hbar}\rho\vartheta < 1 \qquad \vartheta < \frac{\lambda}{\rho}$$

$\boxed{\text{See QG 6}}$

$$\psi = e^{\frac{i}{\hbar} p z} + \sum_{l=0}^{\infty} P_l^{(0)}(\cos\vartheta)\, \frac{a_l}{r}\, e^{\frac{i}{\hbar} p r}$$

$$\overline{P_l^{(0)\,2}} = \frac{1}{2l+1}$$

incident $\frac{p}{l}$ per cm² sec

$$\frac{4\pi}{2l+1} |a_l|^2 \frac{p}{m} = \sigma_l \frac{p}{m}$$

$$\sigma_l = \frac{4\pi}{2l+1} |a_l|^2 \qquad P_l^{0} \text{ (page 10)}$$

$$e^{\frac{i}{\hbar} p z} = \sum_l \varphi_l(r)\, P_l^{(0)}(\cos\vartheta)$$

$$\frac{4\pi}{2l+1} \varphi_l(r) = \int_0^{\pi} e^{\frac{i}{\hbar} p r \cos\vartheta}\, P_l^{(0)}(\cos\vartheta)\, 2\pi \sin\vartheta\, d\vartheta$$

$$\frac{2}{2l+1} \varphi_l(r) = \int_{-1}^{1} e^{\frac{ipr}{\hbar} x}\, P_l^{(0)}(x)\, dx$$

$$2\varphi_0(r) = \frac{2\hbar}{pr} \left(\sin \frac{pr}{\hbar} \right.$$

$$\varphi_0(r) = \frac{\hbar}{pr} \sin \frac{pr}{\hbar}.$$

$$\varphi_l(r) = \sqrt{\frac{\pi}{2}} \sqrt{\frac{\hbar}{pr}}\, i^l (2l+1)\, P_l^{(0)}(\cos\vartheta)\, J_{l+\frac{1}{2}}\left(\frac{pr}{\hbar}\right)$$

$$a(1)\,b(2) \pm b(1)\,a(2) = \psi_{\pm}$$

$$H = \frac{p_1^2 + p_2^2}{2m} + \frac{e^2}{r} + \frac{e^2}{r_{12}} - \frac{e^2}{r_{a1}} - \frac{e^2}{r_{a2}} - \frac{e^2}{r_{b1}} - \frac{e^2}{r_{b2}}$$

$$\int \psi_+^* H \psi_+ \cancel{\Phi}$$

Step a (normalisation)

$$\iint \left(a(1)\,b(2) + b(1)\,a(2)\right)^2 d\tau_1\, d\tau_2 =$$

$$= \int a^2(1)\,d\tau_1 \int b^2(2)\,d\tau_2 + \int b^2(1)\,d\tau_1 \int a^2(2)\,d\tau_2 +$$

$$+ 2 \int a(1)\,b(1)\,d\tau_1 \int a(2)\,b(2)\,d\tau_2$$

$$= 2\left[1 + \beta^2\right] \qquad \beta = \int a(1)\,b(1)\,d\tau_1$$

$$\psi_+ = \frac{a(1)\,b(2) + b(1)\,a(2)}{\sqrt{2(1+\beta^2)}}$$ spins antiparallel

$$\psi_- = \frac{a(1)\,b(2) - b(1)\,a(2)}{\sqrt{2(1-\beta^2)}}$$ spins parallel

Note $\beta = 0$ if e.f. do not overlap. Would be 1 for coincident nuclei

Lattice Containing a Large Number of Cells

The density of neutrons of any given energy in a lattice containing a large number of cells is a function of the position in the lattice. One can arrive at a simple mathematical description of the behavior of such a system by neglecting in first approximation the local variation of such functions due to the periodic structure of the lattice and substituting for the actually inhomogeneous system an equivalent homogeneous system. In this section we shall accordingly simplify the problem by substituting for all densities of neutron values obtained by averaging the actual values over the volume of the cell. The densities will then be represented by smooth functions such as one would expect in a homogeneous uranium-graphite mixture.

Let $Q(x, y, z)$ be the number of fast neutrons produced per unit time and unit volume at each position in the lattice. These neutrons diffuse through the mass and are slowed down. During this process some of the n are absorbed at resonance. Let $q(x, y, z)$ be the number of neutrons per unit time and unit volume which become thermal at the position x, y, z. a is called the "density of the nascent thermal neutrons".

We shall assume that if an original fast neutron is generated at a point O the probability that it becomes thermal at a given place has a Gaussian distribution around O. This assumption may be justified by considering that the diffusion process of slowing down consists of very many free paths. Experimentally one finds that the distribution curve of the nascent thermal neutrons around a point source of fast neutrons is represented only approximately by a Gaussian distribution and formulas have been used in which the actual distribution is described as a super-position of two or three Gaussian curves with different ranges. For the purpose of the present discussion, however, we shall take only one. For each fast neutron produced

84- 2

$$E_+ = \iint \psi_+^* H \psi_+ \, d\tau_1 d\tau_2$$

R = Rydberg energy

Use

$$\left(\frac{p_1^2}{2m} - \frac{e^2}{r_{a1}}\right) a(1) = - R\, a(1)$$

$$\frac{p_1^2}{2m}\, a(1) \doteq \left(\frac{e^2}{r_{a1}} - R\right) a(1)$$

$$\cancel{H\psi^+ = \frac{1}{\sqrt{2(1+\beta^2)}}} \qquad \overbrace{p_1^2/2m} \qquad \overbrace{p_2^2/2m}$$

$$H\, a(1)\, b(2) = \cancel{\frac{e^2}{r_{a1}}} - R + \cancel{\frac{e^2}{r_{b2}}} - R + \frac{e^2}{r} + \frac{e^2}{r_{12}} - \cancel{\frac{e^2}{r_{a1}}} - \frac{e^2}{r_{a2}} - \frac{e^2}{r_{b1}} - \cancel{\frac{e^2}{r_{b2}}}$$

$$= -2R + \frac{e^2}{r} + \frac{e^2}{r_{12}} - \frac{e^2}{r_{a2}} - \frac{e^2}{r_{b1}}$$

$$E_+ = -2R + \frac{e^2}{r} + \frac{1}{2(1+\beta^2)} \iint \big(a(1)\,b(2) + b(1)\,a(2)\big) \times$$

$$\times \left[\left(\frac{e^2}{r_{12}} - \frac{e^2}{r_{a2}} - \frac{e^2}{r_{b1}}\right) a(1)\, b(2) + \left(\frac{e^2}{r_{12}} - \frac{e^2}{r_{a1}} - \frac{e^2}{r_{b2}}\right) b(1)\, a(2)\right] d\tau_1 d\tau_2$$

$$E_\pm = -2R + \frac{e^2}{r} + \frac{1}{1\pm\beta^2} \iint \left(\frac{e^2}{r_{12}} - \frac{e^2}{r_{a2}} - \frac{e^2}{r_{b1}}\right) a^2(1)\, b^2(2)\, d\tau_1 d\tau_2 +$$

$$\pm \frac{1}{1\pm\beta^2} \iint \left(\frac{e^2}{r_{12}} - \frac{e^2}{r_{a2}} - \frac{e^2}{r_{b1}}\right) a(1)\, b(1)\, a(2)\, b(2)\, d\tau_1 d\tau_2$$

uranium and by carbon will be proportional to $N_U \sigma_U \overline{n_U}$ and $N_C \sigma_C \overline{n_C}$ and we will have therefore, instead of Eq. (6) the corrected formula

$$f = \frac{N_U \sigma_U \overline{n_U}}{N_U \sigma_U \overline{n_U} + N_C \sigma_C \overline{n_C}} \tag{7}$$

For practical purposes it is usually sufficiently accurate to calculate $\overline{n_C}$ and $\overline{n_U}$ using the diffusion theory. The approximation is made to substitute the lattice cell by a spherical cell having volume equal to that of the actual cell with the boundary condition that the radial derivative of the density of neutrons vanishes at the surface of the sphere. It is also assumed that the number of neutrons that are slowed down to thermal energies per unit time and unit volume is constant throughout in the graphite part of the cell. This approximation is fairly correct provided the dimensions of the cell are not too large. With these assumptions one finds the following formula for the probability f that thermal neutrons be absorbed by uranium:

$$f = \frac{3\alpha^2}{\alpha^3 - \beta^3} \frac{(1-\alpha)(1+\beta)e^{-(\beta+\alpha)} - (1+\alpha)(1-\beta)e^{\beta-\alpha}}{(\alpha + s - s\alpha)(1+\beta)e^{-\beta+\alpha} - (\alpha + s + s\alpha)e^{\beta-\alpha}} \tag{8}$$

where α and β represent the radius of the lump and the radius of the cell expressed taking the diffusion length in graphite $l = \sqrt{\lambda \Lambda / 3}$ as unit of length. It is further

$$s = \frac{\lambda}{\sqrt{3}\, l} \frac{1+\gamma}{1-\gamma} \tag{9}$$

where γ is the reflection coefficient of the lump for thermal neutrons.

Discussion

take this as zero energy $-2R$ is the energy of the two separate atoms

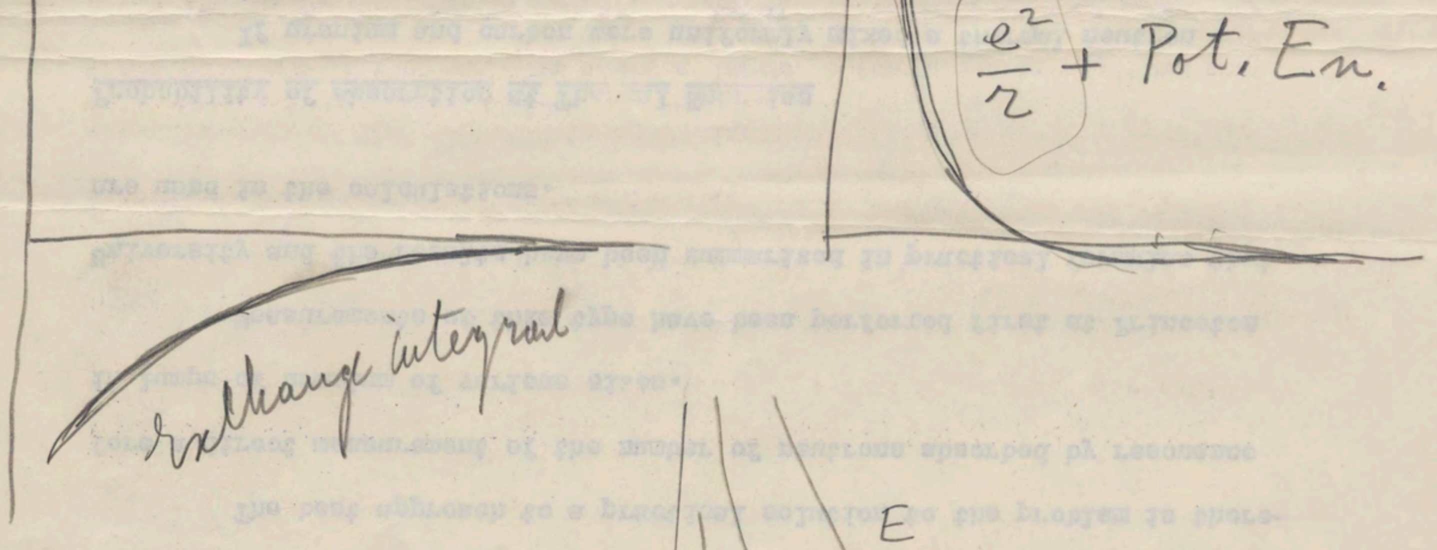
$\frac{e^2}{r}$ " " pot energy of the nuclei

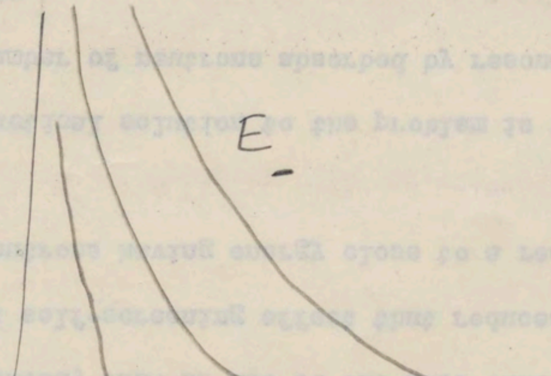
3rd factor apart of β which is small

$\int \rho(1)\rho(2)$ is electrostatic interaction

4th factor is called exchange

and vanishes when $a(1)$, $b(1)$ dont overlap





E_+ solution stable
E_- solution repulsive) In normal H_2 molecule electron spins $\uparrow\downarrow$

The above formula would be very much in error in the case of a lattice of lumps. As already indicated, this is due to the fact that inside a lump there is an important self-screening effect that reduces very considerably the density of neutrons having energy close to a resonance maximum.

The best approach to a practical solution to the problem is therefore a direct measurement of the number of neutrons absorbed by resonance in lumps of uranium of various sizes.

Measurements of this type have been performed first at Princeton University and the results have been summarized in practical formulae that are used in the calculations.

Probability of Absorption at Thermal Energies

If uranium and carbon were uniformly mixed a thermal neutron would have a probability

$$(6) \qquad \frac{N_U \sigma_U}{N_C \sigma_C + N_U \sigma_U}$$

to be absorbed by uranium. In this formula N_C and N_U represent the numbers of atoms of carbon and of uranium per unit volume; and σ_C and σ_U represent the cross-sections of carbon and uranium for thermal neutrons.

More complicated is the case of a lattice distribution of lumps of uranium in graphite, since the density of thermal neutrons throughout the system is not uniform but is large at the places far from the uranium lumps and smaller near and inside the uranium lumps, due to the fact that the absorption of thermal neutrons is much greater in uranium than in graphite. Let $\overline{n_C}$ and $\overline{n_U}$ be the average densities of thermal neutrons in the graphite and in the uranium lumps. The number of thermal neutrons absorbed by

Different approx

Consider as zero approx

1 electron in field of 2 nuclei

$a(1)$ or $b(1)$

or $a(1) + b(1)$

and $a(1) - b(1)$

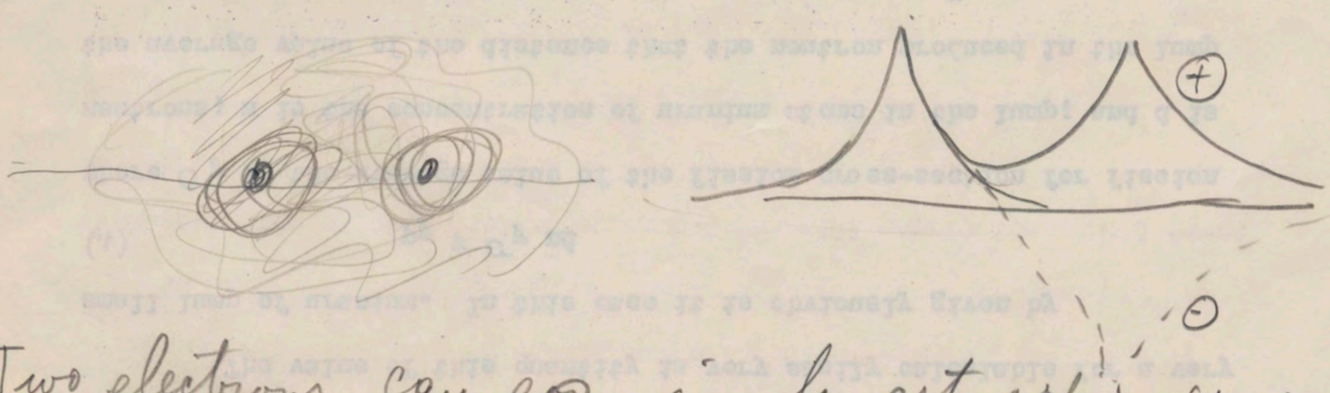
These are degenerate

Solving perturbation problem one finds

$a(1) + b(1)$ is the lowest state

(has no nodes while $a(1) - b(1)$ solution has median plane as a node)

One electron state would b



Two electrons can go in lowest orbit if ↑↓

— UP TO HERE —

Start with Rot 3 next time

Probability of Fission Before Slowing Down

The value of this quantity is very easily calculable for a very small lump of uranium. In this case it is obviously given by

$$p_F = \sigma_F \, nd \qquad (4)$$

where σ_F is the average value of the fission cross-section for fission neutrons; n is the concentration of uranium atoms in the lump; and d is the average value of the distance that the neutron produced in the lump must travel before reaching the surface of the lump. The case of a lump of larger size is more complicated since then multiple collision processes become important and both elastic and inelastic scattering play a considerable role. In particular the last process for a lump of large size effectively slows down the neutrons before the fission threshold of U^{238} and brings them down to an energy level in which they are readily absorbed by the resonance process.

Resonance Absorption

If we had a single atom of uranium in a graphite medium where fast neutrons are produced and slowed down to thermal energy, the probability per unit time of a resonance absorption process of neutrons with energy larger than thermal energy would be given by the following expression:

$$\frac{q\,\lambda}{.158} \int \sigma(E)\frac{dE}{E} \qquad (5)$$

where q is the number of fast neutrons entering the system per unit time and unit volume; λ is the mean free path and $\sigma(E)$ is the resonance absorption cross-section at energy E. The integral must be taken between a low limit just above thermal energy and an upper limit equal to the average energy of the fission neutrons. One will expect that the largest contribution to the integral will be due to the Breit-Wigner peaks of $\sigma(E)$.

Wang solution

Uses Ritz method

Theorem I: if u is arbitrary function normalized to 1 $\int |u|^2 d\tau = 1$

then

$$\int u^* H u \, d\tau \geq \text{minimum e. v}$$

Proof: develop u in e.f. of H

$$H\psi_n = E_n \psi_n$$

$$u = \sum c_n \psi_n$$

$$Hu = \sum c_n E_n \psi_n$$

$$u^* = \sum c_m^* \psi_m^*$$

$$\int u^* H u = \sum_{m,n} c_m^* c_n E_n \underbrace{\int \psi_m^* \psi_n}_{\delta_{mn}} = \sum_n |c_n|^2 E_n$$

Theorem II: if u differs from ψ_1 by inf. of 1st order $\int u^* H u$ differs from E_1 by 2nd order terms

In this case c_2, c_3, , are 1st order

Since $|c_1|^2 + |c_2|^2 + \ldots = 1$ $|c_1|^2 = 1 - |c_2|^2 - \ldots = 1 - \text{2nd order}$

figures for a good lattice will be given as an example. When a neutron is first produced by a fission taking place in a lump of uranium it may have a probability of the order of say 3 percent of being absorbed giving rise to fission before loosing any appreciable amount of energy. In 97 percent of the cases when this does not happen the neutron will initiate its slowing down process and it may either be absorbed by the resonance process during the slowing down or reach thermal energy. The probability of resonance absorption during the slowing down may be of the order of 10 percent so that 87 percent of the original neutrons will be slowed down to thermal energies. Of these perhaps 10 percent may be absorbed by carbon and the remaining 77 percent by uranium. If we assume for the purpose of example that $\nu = 2$, we shall have in one generation the processes summarized by the table.

Probability	Type of Process	Neutrons Produced per Neutron absorbed	Neutrons per Generation by One Neutron
3%	Fast Fission	2	.06
10%	Resonance Absorption	0	0
10%	Absorption by Carbon	0	0
77%	Absorption by Uranium at thermal energies	η	$.77\,\eta$

For the example given the reproduction factor will be, therefore,

$$k = .06 + .77\,\eta \qquad (3)$$

Consequently, a lattice of the type as described would have a reproduction factor larger than 1 provided η is larger than 1.22.

In order to evaluate the reproduction factor one must be able to calculate the probabilities for the various processes mentioned. We shall briefly indicate some points of view which may be used in the practical calculation.

$$\int u^* H u = \underbrace{|c_1|^2 E_1}_{1 + 2\text{nd order}} + \underbrace{|c_2|^2 E_2 + \ldots}_{2\text{nd order}}$$

Hence method: Guess a wave function

Wang's guess was

$$u = \text{Norm. factor} \times \left\{ e^{-\frac{z}{a}(r_{1a}+r_{2b})} + e^{-\frac{z}{a}(r_{2a}+r_{1b})} \right\}$$

a = Bohr radius

z = adjustable parameter

Calculate $\langle r \rangle$

$$\int u^* H u = f(z)$$

Minimise this with resp. to z

	Wang	Experiment
Energy	-2.278 Ryd	-2.326
Mom. of inertia	$.459 \times 10^{-40}$	$.467$
Oscill. freq	$4900\ cm^{-1}$	4360

not restrict ourselves, however, to homogeneous mixtures only, one can try to obtain a more favorable situation by proper arrangement of the geometrical distribution of the two compoennts. This actually is possible to a considerable extent due to the following circumstances. The resonance absorption which is responsible for the loss of neutrons during the slowing down has very sharp cross-section maxima of the Breit-Wigner type. If the uranium, therefore, instead of being spread through the graphite mass is concentrated in rather sizeable lumps, we will expect that the uranium in the interior of a lump will be shielded by a thin surface layer from the action of neutrons with energy close to a resonance maximum. Therefore, the resonance absorption of a uranium atom inside the lump will be much less than it would be for an isolated atom. Of course, self-absorption in a lump reduces not only the resonance absorption but also the thermal absorption of uranium. One can expect theoretically, however, and experiment has confirmed, that at least up to a certain size of lumps the gain obtained by reducing the resonance loss of neutrons over-balances by a considerable amount the loss due to a lesser absorption of thermal neutrons.

The typical structure of a pile is a lattice of uranium lumps embedded in a matrix of graphite. The lattice may be for instance a cubic lattice of lumps or a lattice of rods of uranium. This latter arrangement is slightly less efficient from the point of view of the neutron absorption balance but often presents some practical advantages since it makes the removal of the heat produced by the pile more easy. In the present discussion we shall consider only lattices of lumps.

It is useful to give some typical figures for the probabilities of the various absorption processes. These probabilities of course, are not constant but depend on the details of the structure of the lattice. Average

London Eisenschitz

$\psi_n \varphi_m \qquad E_{nm} = A_n + B_m$

$$H = \frac{2Z\zeta - X\xi - Y\eta}{r^3}$$

$$H_{00,00} = \frac{2 Z_{00}\zeta_{00} \cdots}{r^3} = 0$$

$$\Delta W = -\sum_{n,m} \frac{|H_{00,mn}|^2}{E_{mn} - E_{00}} \approx -\frac{K}{r^6}$$

$$(Z\zeta)_{00,mn} = Z_{0m}\,\zeta_{0n}$$

$$K = \sum_{mn} \frac{|2 Z_{0m}\zeta_{0n} - \cdots|^2}{A_m - A_0 + B_n - B_0}$$

$$Z \sim \frac{1}{e^2}\left(\frac{\hbar^2}{me^2}\right)^4 \quad \cancel{\times 10^{-8}}\,10^{-80}$$

$$10^{-32} \qquad \frac{10^{-33}\, e^4}{\hbar/a}$$

$$\cancel{e^4 x} \quad e^2 x^5 \qquad e^2\left(\frac{\hbar^2}{me^2}\right)^5 \; 10^{-23} \qquad 10^{-40}$$

$$E = E_e + \hbar c\omega\left(v + \tfrac{1}{2}\right) + \hbar c B\left[J(J+1) - \Omega^2\right]$$

ELEMENTARY THEORY OF THE PILE

by E. Fermi
Institute for Nuclear Studies, University of Chicago

I shall attempt in what follows to give an outline of the theory of a chain reacting pile working with natural uranium and graphite.

The results and the methods that I am going to discuss have been obtained partly independently and partly in collaboration by many people who participated in the early development work on the chain reaction. Very important contributions to the theoretical ideas were given by Szilard and Wigner. Many physicists contributed experimental results that helped to lead the way. Among them H. L. Anderson and W. H. Zinn first at Columbia University and later at the Metallurgical Laboratory of the University of Chicago; R. R. Wilson and E. Creutz at Princeton; Allison, Whitaker and V. C. Wilson at the University of Chicago. The production of the chain reaction was finally achieved in the Metallurgical Laboratory directed by A. H. Compton.

Absorption and Production of Neutrons in a Pile

We consider a mass, "the pile", containing uranium spread in some suitable arrangement throughout a block of graphite. Whenever a fission takes place in this system an average number ν of neutrons is emitted with a continuous distribution of energy of the order of magnitude of a million of electron volts. After a neutron is emitted, its energy decreases by elastic collisions with the atoms of carbon and to some extent also by inelastic collisions with the uranium atoms. In the majority of cases the neutrons will be slowed down to thermal energies. This process

$$\dot{a}_n = -\frac{i}{\hbar}\sum_m \mathcal{H}_{mn}\, a_m\, e^{\frac{i}{\hbar}(W_m - W_n)t}$$

$a_m = 1$

$$a_n = -\frac{\mathcal{H}_{mn}}{W_m - W_n}\left(e^{\frac{i}{\hbar}(W_m - W_n)t} - 1\right)$$

$$|a_n|^2 = \frac{|\mathcal{H}_{mn}|^2}{(W_m - W_n)^2}\, 4\sin^2\frac{(W_m - W_n)t}{2\hbar}$$

$$\sum |a_n|^2 = 4|\mathcal{H}_{mn}|^2 \frac{dn}{dW_n}\underbrace{\int \frac{\sin^2\left(\frac{t}{2\hbar}\varepsilon\right)}{\varepsilon^2}}_{\frac{\pi t}{2\hbar}}$$

$$= \frac{2\pi}{\hbar}|\mathcal{H}_{mn}|^2 \frac{dn}{dW_n}\, t$$

Prob of transition per unit time $= \frac{2\pi}{\hbar}|\mathcal{H}_{mn}|^2 \frac{dn}{dW_n}$

$$\mathcal{H}_{mn} = \frac{1}{\Omega}\int V e^{\frac{i}{\hbar}(p'-p)\cdot r}\, d\tau$$

$$\frac{dn}{dW_n} = \frac{p^2\, dp\, d\omega\, \Omega}{8\pi^3 \hbar^3\, v\, dp}$$

$$\frac{\sigma_{d\omega} v}{\Omega} = \frac{2\pi}{\hbar}\frac{1}{\cancel{\Omega}}\left|\int V e^{\frac{i}{\hbar}(p'-p)\cdot r}\, d\tau\right|^2 \frac{p^2\, \cancel{dp}\, d\omega\, \cancel{\Omega}}{8\pi^3\hbar^3 v}$$

$$\sigma_{d\omega} = \frac{p^2/v^2}{4\pi^2\hbar^4}\left|\int V e^{\frac{i}{\hbar}p'-p\cdot r}\, d\tau\right|^2 d\omega$$

requires about 100 collisions with carbon atoms. After the energy of the neutron is reduced to thermal value, the neutron keeps on diffusing until it is finally absorbed. In several cases, however, it will happen that the neutron is absorbed before the slowing down process is completed.

The neutron may be absorbed by either the carbon or the uranium. The absorption cross-section of carbon for neutrons of thermal energy is quite small; its value being approximately $.005 \times 10^{-24}$ cm^2. For graphite of density 1.6 this corresponds to a mean free path for absorption of about 25 m. It is believed that the absorption cross-section follows the 1/v law and consequently the absorption cross-section which is already quite small at thermal energies becomes practically negligible for neutrons of higher energy. It is, therefore, a sufficiently good approximation to assume that absorption by carbon during the slowing down process can be neglected.

The absorption of a neutron by uranium may lead either to fission or to absorption by an (n, γ) process. We shall refer to this last possibility as the process of resonance absorption. The relative importance of fission and resonance absorption in the different energy intervals is not the same. In this respect we can consider roughly three intervals.

a. Neutrons with energy above the fission threshold of U^{238}. We can call these conventionally "fast neutrons". For fast neutrons the most important absorption process is fission which normally takes place in the abundant isotope U^{238}. Resonance absorption is smaller but not negligible.

b. Neutrons of energy below the fission threshold of U^{238} and above thermal energy. We shall refer to these neutrons as "epi-thermal neutrons". For epi-thermal neutrons the most important absorption process

$$\psi \to e^{\frac{i}{\hbar}pz} + \sum_l P_l^{(0)}(\vartheta)\frac{a_l}{r} e^{\frac{i}{\hbar}pr}$$

$$\frac{4\pi}{2l+1} \cancel{\frac{a_l}{a}} \frac{p_m}{m}|a_l|^2 \qquad \overline{P_l^2} = \frac{1}{2l+1}$$

$$\sigma_l = \frac{4\pi}{2l+1}|a_l|^2$$

$$e^{\frac{i}{\hbar}pz} = \sqrt{\frac{\pi}{2}}\sqrt{\frac{\hbar}{pr}}\sum i^l(2l+1)P_l J_{l+\frac{1}{2}}\left(\frac{pr}{\hbar}\right)$$

$$J_{l+\frac{1}{2}}(z) = \frac{2\left(\frac{1}{2}z\right)^{l+\frac{1}{2}}}{\sqrt{\pi}\, l!}\left(1+\frac{d^2}{dz^2}\right)^l \frac{\sin z}{z}$$

$$\to \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4} - \frac{l\pi}{2}\right)$$

$$\psi \to \sum_{l=0}^{\infty}\frac{1}{r}P_l^{(0)}(\vartheta)\left\{e^{\frac{i}{\hbar}pr}\left[a_l - \frac{i}{2}\frac{\hbar(2l+1)}{p}\right] + e^{-\frac{i}{\hbar}pr}(-1)^l\frac{i}{2}\frac{\hbar(2l+1)}{p}\right\}$$

$$\psi = \sum \frac{1}{r}P_l^{(0)}(\vartheta)\, c_l\, v_l(r)$$

$$v_l'' + \left[\frac{2mE}{\hbar^2} - \frac{2mU}{\hbar^2} - \frac{l(l+1)}{r^2}\right]v$$

$$r^{l+1} \leftrightarrow v_l \leftrightarrow A_l \cos\left(\frac{pr}{\hbar} - \frac{l+1}{2}\pi + \beta_l\right)$$

$$a_l = \frac{\hbar(2l+1)}{p}e^{i\beta_l}\sin\beta_l \qquad c_l = (-1)^l e^{i\left(\beta_l - \frac{\pi l}{2}\right)}\frac{\hbar(2l+1)}{pA_l}$$

$$\sigma = \sum \frac{4\pi\hbar^2}{p^2}(2l+1)\sin^2\beta_l$$

Anelastic collisions (Born)

$$\frac{1}{\sqrt{\Omega}} e^{\frac{i}{\hbar}(p \cdot r)} \Psi_i(q)$$

$$\frac{1}{\sqrt{\Omega}} e^{\frac{i}{\hbar}(p' \cdot r)} \Psi_f(q)$$

$$\mathcal{H} = \frac{1}{\Omega} \int V(q, r) \Psi_i^* \Psi_f e^{\frac{i}{\hbar}(p' - p \cdot r)}$$

$$\frac{\Omega}{h^3}$$

$$\Delta \psi + \frac{2m}{\hbar^2}(E - V)\psi = 0$$

$$V = V_0 + \Sigma V_n$$

$$\Delta \psi - \frac{2m}{\hbar^2} V_n \psi = 0 \qquad \psi = \frac{v}{r}$$

$$v'' = \frac{2m}{\hbar^2} V_n v = 0$$

$$\int V_n \psi \, d\tau = 4\pi \int_0^\rho r V_n v \, dr$$

$$= \frac{2\pi \hbar^2}{m} \int_0^\rho r v'' dr = \frac{2\pi \hbar^2}{m} \left[\rho v'(\rho) - v(\rho)\right]$$

$$v(r) = \bar{\Psi}\left(r + \frac{a}{\cancel{\quad}}\right)$$

$$v'(r) = \bar{\Psi}$$

$$\int V_m \psi \, d\tau = -\frac{2\pi\hbar^2}{m} a \bar{\Psi}$$

$$\overline{\Sigma V_m \psi} = -\frac{2\pi\hbar^2 a}{m} n \bar{\Psi}$$

$\cancel{\frac{2m}{\hbar^2} E}$ Anelastic collisions

Born $\quad \sigma_{d\omega} = \dfrac{m^2}{4\pi^2\hbar^4} \dfrac{p\, v'}{v} \left| \int V(q, r) e^{\frac{i}{\hbar}(p \ldots)} \right|^2$

Capture process (Born)

$$\frac{1}{\sqrt{\Omega}} e^{\frac{i}{\hbar}(pr)} \psi_i(q) \qquad \frac{1}{\sqrt{\Omega}} e^{\frac{i}{\hbar}(q,\rho)} \psi_f(r, q)$$

$$\cancel{\mathcal{H} = \frac{1}{\sqrt{\Omega}} \iint e^{\frac{i}{\hbar}(pr)} \psi_i(q) \psi_f^*(q, r) \, dq}$$

$$\cancel{d_f = -\frac{i}{\hbar}} \qquad \mathcal{H} = \frac{1}{\Omega} \iint V(r, \rho, q) e^{\frac{i}{\hbar}[(pr) - (q\rho)]} \psi_f^* \psi_i$$

$$\frac{p'^2}{v'}$$

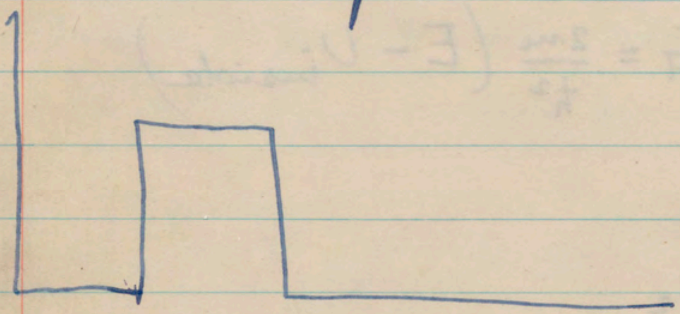
$$\frac{\Omega}{8\pi^3\hbar^3} m^2 v' \, d\omega$$

~~Capture process~~

Sp heat

$$\overline{L} = \frac{3}{40}\left(\frac{6}{\pi}\right)^{3/2} \frac{h^2 n^{2/3}}{m\, g^{2/3}} + \frac{2^{1/3}\pi^{8/3} g^{2/3}}{3^{2/3}} \frac{m \kappa^2 T^2}{h^2 n^{2/3}}$$

Compound nucleus



$\cancel{A + B\Delta E^2 \qquad B \sim \frac{}{\hbar\nu}}$

88

$$A^2 = \frac{G}{\sqrt{\gamma g(\rho)}} e^{-2\int_\rho^b \sqrt{g}\,dr}$$

$$B^2 = \frac{\cancel{16\pi^4} m^2}{\hbar^4} \frac{\rho^2 \sqrt{g(\rho)}}{G\sqrt{\gamma}} e^{2\int_\rho^b \sqrt{g}\,dr}$$

$$g = \frac{2m}{\hbar^2}\left(U_{barrier} - E\right)$$

$$\gamma = \frac{2m}{\hbar^2}\left(E - U_{\cancel{interior}\ outside}\right)$$

$$\frac{m^2 \cancel{\rho^2}}{\hbar^4 \cancel{\gamma\ldots}} \quad \hbar^2\gamma^2$$

$$G = \frac{2m}{\hbar^2}\left(E - U_{inside}\right)$$

$$\tau = \frac{\hbar B}{2A}$$

$$\left(\frac{\text{amp outside}}{\text{amp inside}}\right)^2 = A^2 + B^2\,\delta E^2$$

$$A \sim e^{-\int_\rho^b \sqrt{g}\,d\rho}$$

$$B \sim \frac{1}{\hbar\gamma} e^{\int_\rho^b \sqrt{g}\,d\rho}$$

$$\tau \sim \frac{\hbar}{2}\frac{1}{\hbar\gamma} e^{2\int_\rho^b \sqrt{g}\,dr}$$

$$r^{l+1} \longleftrightarrow v_l(r) \longleftrightarrow A_l \cos\left(\frac{pr}{\hbar} - \frac{l+1}{2}\pi + \beta_l\right)$$

$$\psi \to e^{\frac{i}{\hbar}pz} + \sum_l P_l^{(0)} \frac{a_l}{r} e^{\frac{i}{\hbar}pr}$$

$$\psi = \sum \frac{1}{r} P_l^{(0)} c_l v_l(r)$$

$$a_l = \frac{\hbar(2l+1)}{p} e^{i\beta_l} \sin\beta_l$$

$$c_l = (-1)^l e^{i\left(\beta_l - \frac{l\pi}{2}\right)} \frac{\hbar(2l+1)}{p A_l}$$

$$\dot{a} = -\frac{i}{\hbar} \mathcal{H} a_n e^{\frac{i}{\hbar}(w_0 - w_n)t} - \frac{a}{\tau}$$

$$\cancel{a = \frac{\mathcal{H} a_n}{w_0 - w_n}\left(e^{\frac{i}{\hbar}(w_0 - w_n)t}\right.}$$

$$\dot{a} + \frac{a}{\tau} = -\frac{i}{\hbar} \mathcal{H} a_i e^{\frac{i}{\hbar}(w_0 - w_i)t}$$

$$-\frac{i}{\hbar} \mathcal{H} \sum a_n e^{\frac{i}{\hbar}(w_0 - w}$$

$$\dot{a}_0 = -\frac{i}{\hbar} a_1 \mathcal{H} e^{\frac{i}{\hbar}(W_0 - W_1)t} -$$

$$-\frac{i}{\hbar} \sum a_n \mathcal{H}_{n0} e^{\frac{i}{\hbar}(W_0 - W_n)t}$$

$$\dot{a}_n = -\frac{i}{\hbar} \cancel{\Sigma} a_0 \mathcal{H}_{0n} e^{\frac{i}{\hbar}(W_n - W_0)t}$$

$$\cancel{a_n \; i}$$

$$\dot{a}_0 + \frac{a_0}{\tau} = -\frac{i}{\hbar} a_i \mathcal{H}_{i0} e^{\frac{i}{\hbar}(W_0 - W_i)t}$$

$$a_0 = \frac{-\frac{i}{\hbar} a_i \mathcal{H}_{i0} e^{\frac{i}{\hbar}(W_0 - W_i)t}}{\frac{1}{\tau} + \frac{i}{\hbar}(W_0 - W_i)}$$

$$\dot{a}_n = -\frac{1}{\hbar^2} a_i \mathcal{H}_{i0} \frac{\mathcal{H}_{0n} e^{\frac{i}{\hbar}(W_n - W_i)t}}{\frac{1}{\tau} + \frac{i}{\hbar}(W_0 - W_i)}$$

$$a_n = \frac{i}{\hbar} a_i \mathcal{H}_{i0} \mathcal{H}_{0n} \frac{e^{\frac{i}{\hbar}(W_n - W_i)t} (-1)}{\left[\frac{1}{\tau} + \frac{i}{\hbar}(W_0 - W_i)\right](W_n - W_i)}$$

$$= -a_0 \frac{e^{\frac{i}{\hbar}(W_n - W_0)t} - e^{\frac{i}{\hbar}(W_i - W_0)t}}{W_n - W_i} \mathcal{H}_{0n}$$

$$\frac{1}{t} = \frac{i}{\hbar} \sum \frac{\cancel{e^{\frac{i}{\hbar}(W_n - W_0)t}}\; 1 - e^{\frac{i}{\hbar}(W_i - W_n)t}}{W_n - W_i} |\mathcal{H}_{0n}|^2$$

$\cancel{\frac{1}{h}}$ $$\int_{-\infty}^{\infty} \frac{\sin qx}{x}\, dx = \pi$$

$\cancel{1-\cos}$ $$\int_{-\infty}^{\infty} \frac{\sin^2 kx}{x^2}\, dx = \pi k$$

$$\int_{-\infty}^{\infty} \frac{1-\cos kx}{x} = 0$$

~~Gamow~~

Virtual states

Virtual state (A) amplitude

Continuum states

$$\frac{2\pi}{\hbar} |\mathcal{H}|^2 n$$

$$\mathcal{H} = \frac{H}{\sqrt{\Omega}} \qquad n = \frac{4\pi\, p^2 dp\, \Omega}{8\pi^3 \hbar^3\, v\, dp} = \frac{\Omega}{2\pi^2 \hbar^3} \frac{p^2}{v}$$

$$\frac{1}{\tau} = \frac{|H|^2}{\pi \hbar^4} \frac{p^2}{v}$$

$$\Gamma = \frac{|H|^2}{\pi \hbar^4} \frac{p^2}{v} \qquad \frac{1}{\tau} = \sum \Gamma$$

$$\cancel{A = -\frac{i}{\hbar} a_1 \frac{H}{\sqrt{\Omega}}}$$

92

$$\dot{A} = -\frac{i}{\hbar} a_{\cancel{t}} \sqrt{n}\, H_a e^{\frac{i}{\hbar}(w_A - w_a)t}$$

$$- \frac{i}{\hbar} \sum_s b_s \frac{H_b}{\sqrt{\Omega}} e^{\frac{i}{\hbar}(w_A - w_s)t}$$

$$- \frac{i}{\hbar} \sum_{\cancel{t}} c_{\cancel{t}} \frac{H_c}{\sqrt{\Omega}} e^{\frac{i}{\hbar}(w_o - w_r)t}$$

$$\dot{A} + \frac{1}{2\tau} A = -\frac{i}{\hbar} a \sqrt{n}\, H_a e^{\frac{i}{\hbar}(w_o - w_a)t}$$

$$A = \frac{-\frac{i}{\hbar} a \sqrt{n}\, H_a e^{\frac{i}{\hbar}(w_A - w_a)t}}{\frac{1}{2\tau} + \frac{i}{\hbar} H_a (w_o - w_a)}$$

$$\dot{b}_s = -\frac{i}{\hbar} A \frac{H_b^*}{\sqrt{\Omega}} e^{\frac{i}{\hbar}(w_s - w_A)t}$$

$$b_s = + \frac{i}{\hbar} \cancel{a} A \sqrt{n}\, H_a \frac{H_b^*}{\sqrt{\Omega}} \frac{e^{\frac{i}{\hbar}(w_s - w_a)t} -}{\frac{1}{2\tau} + \frac{i}{\hbar} H_a (w_o - w_a)}.$$

$$b_s = \frac{\dot{b}_s}{\frac{i}{\hbar}(w_s - w_a)}$$

$\dot{A}$

$$e^{\frac{i}{\hbar}(w_s - w_a)} - 1$$

$$\cancel{A = \frac{i}{\hbar}}$$

$$\dot{A} + \frac{A}{2\tau} = -\frac{i}{\hbar}\sqrt{n}\, H_a\, e^{\frac{i}{\hbar}(W_A - W_a)t}$$

$$A = -\frac{\frac{i}{\hbar}\sqrt{n}\, H_a\, e^{\frac{i}{\hbar}(W_A - W_a)t}}{\frac{1}{2\tau} + \frac{i}{\hbar}(W_A - W_a)}$$

$$\frac{|A|^2}{n} = \frac{|H_a/\hbar|^2}{\frac{1}{4\tau^2} + \frac{1}{\hbar^2}(W_A - W_0)^2} = \cancel{\frac{\pi\hbar^3 v}{p^2}}$$

Discuss width of resonance

$$\dot{b}_s = -\frac{i}{\hbar} A \frac{H_b}{\sqrt{\Omega}} e^{\frac{i}{\hbar}(W_b - W_A)t}$$

$$|H_a^2| = \pi \hbar^4 \frac{v}{p^2} \Gamma_a$$

$$\Gamma_b |A|^2 = \frac{n v \hbar^2}{p^2} \frac{\Gamma_a \Gamma_b}{\frac{\Gamma^2}{4} + \frac{(W_a - W_0)^2}{\hbar^2}} = n v\, \sigma(a, b)$$

$$\sigma(a, b) = 4\lambda\!\!\!^{-2} \frac{\Gamma_a \Gamma_b}{\Gamma^2 + \frac{4}{\hbar^2}(W_a - W_0)^2}$$

~~1000 × 1000~~

$$\frac{1000}{10^6}$$

$$A = -\frac{\frac{i}{\hbar}\sqrt{n}\; H_a\; e^{\frac{i}{\hbar}(W_A - W_0)t}}{\frac{\Gamma}{2} + \frac{i}{\hbar}(W_A - W_0)}$$

$$\cancel{\dot{a}}_s = -\cancel{\frac{i}{\hbar}}\frac{H_a^*}{\sqrt{\Omega}}\left(\frac{-\frac{i}{\hbar}\sqrt{n}\,H_a}{\frac{\Gamma}{2}+\frac{i}{\hbar}(W_A - W_0)}\right)\frac{e^{\frac{i}{\hbar}(W_s - W_0)t}-1}{W_s - W_0}$$

$$\cancel{\dot{b}}_{\cancel{k}} = -\cancel{\frac{i}{\hbar}}\frac{H_b^*}{\sqrt{\Omega}}\left(\frac{-\frac{i}{\hbar}\sqrt{n}\,H_a}{\frac{\Gamma}{2}+\frac{i}{\hbar}(W_A - W_0)}\right)\frac{e^{\frac{i}{\hbar}(W_r - W_0)t}-1}{W_r - W_0}$$

$$|b_r|^2 = \frac{1}{\hbar^2}\,\frac{|H_a|^2|H_b|^2\, n}{\Omega\left(\frac{\Gamma^2}{4}+\frac{1}{\hbar^2}(W_A - W_0)^2\right)}\;\frac{4\sin^2\left(\frac{W_r - W_0}{2\hbar}t\right)}{(W_r - W_0)^2}$$

$$\sum|b_r|^2 = \frac{\cancel{4}}{\cancel{\hbar}^2}\,\frac{|H_a|^2|H_b|^2\, n}{\left(\frac{\Gamma^2}{4}+\frac{1}{\hbar^2}(W_A - W_0)^2\right)}\;\frac{\pi t}{\cancel{2}\hbar}\;\frac{p_b^2}{v_b}\;\frac{1}{\cancel{2}\pi^2\hbar^3} = v_0 t\,\sigma_b\, n$$

$$\cancel{\sigma_b t = \frac{1}{v_0} t\, \frac{1}{\pi\hbar}}\qquad \boxed{\sigma_b = \frac{1}{v_0}\,\frac{1}{\pi\hbar^6}\,\frac{p_b^2}{v_b}\,\frac{|H_a|^2\,|H_b|^2}{\frac{\Gamma^2}{4}+\frac{(W_A - W_0)^2}{\hbar^2}}}$$

$$\cancel{\frac{\Gamma}{2} A}$$

$$\Gamma = \frac{1}{\pi\hbar^4}\left\{|H_a|^2\frac{p_0^2}{v_0} + |H_b|^2\frac{p_b^2}{v_b}\right\}$$

$$\cancel{\frac{\Gamma}{2}\hbar}$$

$$\sigma(n,\gamma)\,\cancel{\sigma} = \frac{1}{v}\,\hbar^2\,\frac{\pi\,\cancel{\hbar^4}}{\cancel{\pi\hbar^6}}\;\frac{\Gamma_\gamma\,\cancel{\pi\hbar^4}\,\frac{v_0}{p_0^2}\,\Gamma_n}{\frac{\Gamma^2}{4}+\frac{\Delta W^2}{\hbar^2}}$$

Hornbostel

Rh

In

$$\sim 1\,eV \begin{cases} \sigma_{max} = \cancel{4}8200 \times 10^{-24} & \hbar\Gamma = .13\,eV \quad \hbar\Gamma_n = \cancel{3.4} \times 10^{-4} \\ \sigma_{max} = \cancel{4}6000 \times 10^{-24} & \hbar\Gamma = .07 \quad \hbar\Gamma_n = 1 \times 10^{-3} \end{cases}$$

$$\cancel{\sigma(n,\gamma) = 4\pi\lambda^2 \frac{\Gamma_n\Gamma_\gamma}{\Gamma^2} \frac{1}{1+\frac{4}{\hbar^2\Gamma^2}(W_n-W_0)^2}}$$

$$\begin{cases} \sigma(n,\gamma) = \frac{v_0}{v} 4\pi \lambda_0^2 \frac{\Gamma_n\Gamma_\gamma}{\Gamma^2} \frac{1}{1+\frac{4}{\hbar^2\Gamma^2}(W_n-W_0)^2} \\ \sigma_{max}(n,\gamma) = 4\pi\lambda_0^2 \frac{\Gamma_\gamma\Gamma_n}{\Gamma^2} \end{cases}$$

$1\,eV \qquad \lambda = \frac{1.052\times10^{-27}}{2.3\times10^{-18}} = 4.5\cancel{6}\times10^{-10}$

$\frac{p^2}{2m} = 1.6\times10^{-12} \qquad p^2 = 5.31\times10^{-36} \qquad p = 2.3\times10^{-18}$

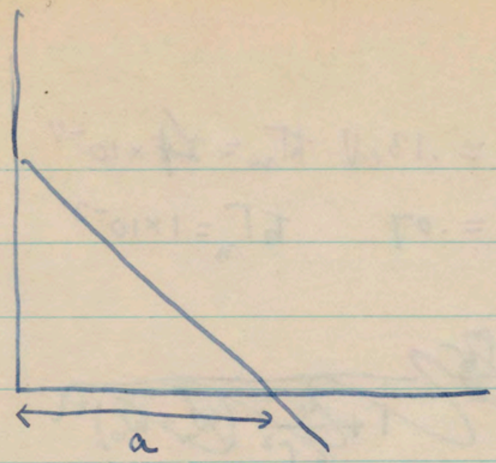
$4\pi\lambda^2 = 2.6\times10^{-18}$

$10^{-\cancel{22}}$ $\qquad \frac{2.6\times10^{-18}}{\sqrt{1\times0.025}} \qquad \frac{.13\times4\times10^{-4}}{1^2} \qquad \frac{0.07\times1\times10^{-3}}{1^2}$

850

$\frac{2.3\times10^{+4}\times10^{-\cancel{6}}}{\cancel{2.3\times10^{-18}}} = 2.3\times10^{-2}$

96



$$u'' + \frac{\cancel{2}m}{\hbar^2} E u = 0 \qquad E = -W$$

$$e^{-\sqrt{\frac{\cancel{2}m}{\hbar^2}}\, W r}$$

$$\sqrt{\frac{\cancel{2}m W_0}{\hbar^2}}\, a = 1 \qquad \boxed{a = \frac{\hbar}{\sqrt{m W_0}}}$$

$$\frac{m v^2}{4} = E = \frac{W}{2}$$

$$u'' + \frac{m W}{2\hbar^2} u = 0$$

$$u = \frac{\sin\left(\sqrt{\frac{mW}{2\hbar^2}}\, r + \beta\right)}{\sin\beta}$$

$$-\frac{1}{a} = \sqrt{\frac{mW}{2\hbar^2}}\, \frac{\cos\beta}{\sin\beta} \qquad \sigma = \frac{4\pi\hbar^2}{p^2}\sin^2\beta$$

$$-\sqrt{\frac{m w_0}{\hbar^2}} = \sqrt{\frac{m w}{2\hbar^2}} \operatorname{cotg}\beta$$

$$\operatorname{tg}\beta = -\sqrt{\frac{w}{2w_0}}$$

$$\sin^2\beta = \frac{\frac{w}{2w_0}}{1+\frac{w}{2w_0}}$$

$$\sigma = \frac{\cancel{8}\pi\hbar^2}{m}\;\frac{\cancel{w}/2w_0}{1+\frac{w}{2w_0}}$$

$$\sigma = \frac{4\pi\hbar^2}{m}\left[\frac{3}{4}\frac{1}{|w_0|+\frac{w}{2}} + \frac{1}{4}\frac{1}{|w_1|+\frac{w}{2}}\right\}$$

$w_0 = 2.17$ MeV

$w = 0 \qquad \sigma = 20\times 10^{-24}$

$$\frac{4\pi\hbar^2}{m\,(1\,\mathrm{MeV})} = \frac{4\pi \times 1.054^2\times 10^{-54}}{1.67\times 10^{-24}\times 1.6\times 10^{-6}} = 5.2\times 10^{-24}$$

$$\frac{3}{4\times 2.17} + \frac{1}{4|w_1|} = 3.85$$

$$\begin{array}{r} 3.85 \\ .334 \\ \hline \cancel{7.50} \\ 3.51 \end{array}$$

~~$w_1 = 0.034$ MeV~~

$|w_1| = 0.071$ MeV

$a \ll R$

$\rho \ll \cdot R$ $\qquad R \ll \lambda$

10^{-12} $\qquad 10^{-9}$ $\qquad X, Y, Z$ H

$x\ y\ z$ neut

$$-\frac{\hbar}{i}\dot{\psi} = -\frac{\hbar^2}{2M}\left(\frac{\partial^2\psi}{\partial x^2} + \dots + \frac{\partial^2\psi}{\partial X^2} + \dots\right) + U(X, Y, Z)\psi + g(r)\psi$$

$$\overline{\psi} = \frac{1}{\frac{4}{3}\pi R^3}\underset{\cdot R \text{ around } xyz}{\iiint}\psi(\xi\,\eta\,\zeta\,X\,Y\,Z)\,d\xi\,d\eta\,d\zeta$$

$$-\frac{\hbar}{i}\dot{\overline{\psi}} = -\frac{\hbar^2}{2M}\left(\frac{\partial^2\overline{\psi}}{\partial x^2} + \dots + \frac{\partial^2\overline{\psi}}{\partial X^2} + \dots\right) + U\overline{\psi} + \overline{g(r)\psi}$$

$$\overline{g(r)\psi} = -\frac{4\pi\hbar^2 a}{M}\delta_R(r)$$

$$\sigma_{mn,d\omega} = 4a^2 \frac{p}{p_0} d\omega \left| \int u_m^* u_n e^{\frac{i}{\hbar}(\vec{p}-\vec{p_0}, x)} d\tau \right|^2$$

Strong binding

$$\sigma = 16\pi a^2$$

No binding: $u_n = e^{\frac{i}{\hbar}(p_n x)} \frac{1}{\sqrt{\Omega}}$ $\quad u_m = \frac{1}{\sqrt{\Omega}}$

$$\vec{p_0} = \vec{p_n} + \vec{p} \qquad p_0^2 = p_n^2 + p^2$$

$$|p| = |p_0| \cos\vartheta$$

$$\sigma_{d\omega} = 4a^2 \cos\vartheta \, d\omega$$

$$\sigma = 4a^2 \int_0^{\pi/2} \cos\vartheta \; 2\pi \sin\vartheta \, d\vartheta = 4\pi a^2$$

100

a) The state of the system is defined by the knowledge of the wave function.
~~b)~~ Wave functions
b) ~~Given a physical magnitude we~~
~~A~~ Hermithian operator corresponding to a physical magnitude A
~~If~~

$$\cancel{A\psi = a\psi}$$

~~follows~~

$$\cancel{A = a}$$

~~if not only the prob dist of the~~
The only allowed values of A are its proper values.
If ψ is a proper function of A namely

$$A\psi = a\psi$$

a measurement of A gives certainly

$$A = a$$

If not develop ψ in proper functions of A

$$\cancel{A}\ \psi = \sum c_n \psi_n$$

if everything is normalised

$$|c_n|^2$$

is the probability of $A = a_n$
If measurements give

$$A = a_n$$

the measurement affects the state bringing it to

$$\psi = \varphi_n$$

c) ψ varies with time according to the Schrödinger equation

$$H\psi = -\frac{\hbar}{i}\frac{\partial \psi}{\partial t}$$

Time derivative

$$\cancel{A\psi(t)}$$

$$A\left(\psi(0) + t\,\frac{\partial \psi}{\partial t}\right)$$

$$\left(1 - \frac{i\tau}{\hbar}H\right)\psi(t) = \psi(t+\tau)$$

$$\left(1 + \frac{i\tau}{\hbar}H\right)\psi(t+\tau) = \psi(t)$$

$$A\psi(t+\tau) = \varphi(t+\tau)$$

$$A\left(1 - \frac{i\tau}{\hbar}H\right)\psi(t) = \left(1 - \frac{i\tau}{\hbar}H\right)\varphi(t)$$

$$\psi(t+\tau) = \sum c_n^{(t)}\varphi_n$$

$$A u_n = a_n u_n$$

$$\cancel{\psi(t+\tau)}$$

$$\psi(\tau) = \sum c_n u_n$$

$$A\psi(\tau) = \sum c_n a_n u_n$$

$$A\left(1 - \frac{i\tau}{\hbar}H\right)\psi(0) = \sum c_n a_n u_n$$

$$= A \sum c_n u_n$$

$$\left(1 + \frac{i\tau}{\hbar}\cancel{H}\right) A \left(1 - \frac{i\tau}{\hbar}H\right)$$

$$(1+\ \ )A(1-\ \ )\psi_0 = (1+\ \ )\sum c_n a_n u_n$$

$$= \sum c_n a_n (1+\ \ ) u_n$$

$$(1+\ \ )A(1-\ \ )(1+\ \ )u_n = a_n(1+\ \ )u_n$$

A $\quad a_1 \quad a_2 \cdots a_n$

$\quad u^{(1)} \quad u^{(2)} \quad u^{(n)}$

Transformation matrix

$T \qquad t_{ik} \qquad T^{-1} \; \dfrac{\text{minor}\,(t_{ki})}{|T|}$

$$TT^{-1} = T^{-1}T = 1$$

TAT^{-1} has p. val. $a_1 \quad a_2 \dots \quad a_n$
prop funct $Ta_1 \quad Ta_2 \dots Ta_n$

~~lues~~ proof
$$\boxed{\begin{aligned} A u^{(r)} &= a_r u^{(r)} \\ TAT^{-1} T u^{(r)} &= a_r T u^{(r)} \end{aligned}}$$

Unitarian matrix: $\tilde{T} T = 1 \quad \tilde{T} = T^{-1}$

if ~~[illegible]~~ $\sum_K u_K^{(r)*} u_K^{(s)} = \delta_{rs}$

~~f~~follows $\sum_K u_K^{(r)*} \tilde{T} T u_K^{(s)} = \delta_{rs} \quad \tilde{T} T = 1$

~~Matrix $u_K^{(s)}$ is unitarian~~

~~$t_{sK} (t)_{Kl} = \delta_{sl} \quad u_K^{(s)} \quad u_K^{(s)} = t_{sK}$~~

~~$\sum_K u_K^s u_K^{l*} = 1$~~

~~$T_{sK} = u_K^{(s)}$ is a unitarian matrix~~

~~$\tilde{T}_{Ks} = u_K^{(s)*} \quad u_K \quad u_K$~~

~~$\tilde{T} A T \quad \tilde{T} A T \quad \tilde{T}$~~

~~$\tilde{T} A T \quad f u_1 \quad \tilde{T} f = u_K^{(s)*}$~~

104

Matrix

$$T_{ks} = u_k^{(s)}$$

is unitarian.

Proof

$$\tilde{T}_{sk} = u_k^{(s)*}$$

$$(\tilde{T}T)_{sr} = \sum_k \tilde{T}_{sk} T_{kr} = \sum_k u_k^{(s)*} u_k^{(r)} = \delta_{rs}$$

$$T_{ks} = u_k^{(s)}$$

Transforms A in the diagonal matrix

$$\tilde{T}AT = \begin{vmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & \\ & & \ddots & \\ 0 & & & a_n \end{vmatrix}$$

Proof

$$\tilde{T}AT = u_k^{(s)*} a_{kl} u_l^{(r)}$$

$$= a_r u_k^{(s)*} u_k^{(r)} = a_r \delta_{rs}$$

The proper functions are

$$\tilde{T} u^{(s)} = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$ at pos s

$$(\tilde{T} f) = u_k^{s*} f_k$$ = coeff of expansion

Meaning of $T\tilde{T} = 1$

$$u_k^{(s)} u_j^{(s)*} = \delta_{kj}$$

or orthogonality for summation on the upper index

Case of continuum

$$u_s(x)$$

$$\int u_s^*(x)\, u_r(x)\, dx = \delta_{rs}$$

$$\sum_s u_s^*(x')\, u_s(x) = \delta(x-x')$$

Proof: Develop $\delta(x-x')$ in a series of p.f.

Average value of an operator

$$\tilde{\psi} A \psi \quad \text{or} \quad \int \psi^*(A\psi)\, d\tau$$

35

Perturbations with degeneracy or quasi-degeneracy

(See pages 35–36)

$$u_1^0 \; u_2^0 \; u_3^{(0)} \dots u_g^{(0)} \;\Big|\; u_{g+1}^{(0)} \; u_{g+2}^{(0)} \dots$$

$$\sum_1^g c_i u_i^{(0)} + \sum_{g+1}^{\infty} c_\alpha u_\alpha^{(0)} = u \qquad \begin{cases} c_\alpha & \text{small} \\ c_i & \text{large} \end{cases}$$

$$\cancel{(H_0 + \mathcal{H})u = \sum_1^g c_i w_i^0 u_i^0 + \sum_1^g c_i \mathcal{H} u_i^{(0)} +}$$

$$\cancel{+ \sum_{g+1}^{\infty} c_\alpha w_\alpha^{(0)} u_\alpha^{(0)} = w\left(\sum c_i u_i^0 + \sum c_\alpha u_\alpha^0\right)}$$

$$H = H_0 + \mathcal{H}$$

$$\sum_1^g c_i H u_i^0 + \sum_{g+1}^{\infty} c_\alpha w_\alpha^0 u_\alpha^0 = w \sum_1^g c_i u_i +$$

$$+ w^{(0)} \sum_g^{\infty} c_\alpha u_\alpha^0$$

$$w c_i = \sum_1^g c_K H_{iK}$$

Secular equation for w with g solutions

$$c_\alpha (w_\alpha^0 - w^0) + \sum c_i H_{\alpha i} = 0$$

no infinity

$HA - AH = 0$ ~~tw~~

$H_0 A - A_0 H_0 = 0$

Classes with respect to A do not mix

~~Perturbation~~

(4) Discuss the theory of perturbations ~~problem~~ for the case that ~~when~~ the unperturbed state is degenerate or quasi-degenerate

~~Calculate the life-time of the 2p-state of hydrogen~~

3 Discuss the meaning of the time derivative of an operator

~~The energy operator of a~~

~~An~~ an electron can move ~~is movable~~ on a ~~segment~~

2 of length $\underline{a}^0$ ~~fu~~. Find the probability of spontaneous transition from the second to the first energy level.

Find whether or not

1 ~~Which~~ of the following operators are hermitian:

$$\cancel{\frac{d}{dx}},\ \frac{d^2}{dx^2},\ \frac{d^3}{dx^3},\ x^2 p_x + x p_x^2,$$

$$x p_x^2 + p_x^2 x$$

108

Outline of Quantum Mechanics Course (Chicago – Fall 1947 + Winter 1948)

Oct 1 × 1 Historical Resume up to 1925

(10)1 × 2 Mechanics – Optics analogy

× 3 Time dependent + time independent Schrodinger equation

× 4 Generalizations to simple cases.

× 5 Examples. Point on closed line + rotator with fixed axis – Normalization integral

× 6 Density of probability + Density of current

× 7 The linear oscillator

2 × 8 Orthogonality in one dimension – Point on a spherical surface – Resume on spherical harmonics – Reduction of the Schrodinger equation ~~of motion~~ of a point in a central field.

× 9 Hydrogenlike atoms: negative energy levels.

× 10 Positive energy levels – Expression of e.f. in terms of Laguerre polinomials,

× 11 Degeneracy of the hydrogen levels. Notation s, p, d,...

× 12 Potential energy $-\frac{Ze^2}{r}\left(1+\frac{\beta}{r}\right)$ and qualitative discussion of one electron problem approximation of heavy atoms.

13× WKB method ×

8 14× Orthogonality in three dimensions −1

15× Degenerate eigenfunctions ×

16× Operators – Linear operators – Matrices ×

17× ~~Fac~~ function as vectors – Scalar product add Hamilton

Sum Difference + Product of operators

18× Sum, difference + products of matrices.

19× Eigenvalues & e. f of operators – Secular equation for the e. v of a matrix – Hermitian matrices have real e. v.

20× Geom. interpretation of e. v + e. f. – Hermitian operators of continuous fields.

21× Development in e. f. – Geom. discussion Functions of Operators – State of a system

×22 Examples on $\Delta p \Delta q \sim h$ and $\Delta W \times T \sim h$ ×

×23 Wave interpretation of uncertainty principle ×

×24 Operator $p = \frac{\hbar}{i}\frac{d}{dx}$ ×

×25 Operator x – δ-function – Operators corresponding to physical quantities.

Dec 1 ×26 Schrödinger function as state function

×27 Commutation and its significance

28 Properties of angular momentum operators ×

29 Time independent perturbation theory ×

110

30 Discussion on preceeding lecture ×
31 Stark effect on Hydrogen atom ×
32 Zeeman effect without spin
33 Discussion of time dependent Schrödinger equation
34 Time dependent perturbation theory

Winter Quarter

Jan 5 1948 1 Discussion of time dependent perturb. theory
2 Absorption of light (From perturbation by electric field of wave)
3 Emission of light (From A's and B's)
4 Selection rules $l \to l \pm 1$ $m \to m \pm_0^1$, Quadrupole radiation
5 Spin of the electron - Generalities
6 The Pauli operators
7 Electron in central field - Unperturbed problem
8 Perturbation matrix.
9 Perturbation of energy levels
10 Interpretation with vectors - Description of the Hydrogen fine-structure.
11 Anomalous Zeeman effect
12 Landé g-formula with vector model
Feb 2 13 Many (two) particle systems.

111

\- Systems with two equal particles - Symmetry & antisymmetry.

14 Antisymmetric functions & Pauli principle

15 Exchange integral splitting of terms of 2-electron system

16 Discussion of preceding case

17 The helium spectrum

18 Selection rules - Even and odd states

19 Electron orbits & periodic table

20 Spectra of alcali atoms & alcaline earths

21 Spectra of earths - Various types of couplings.

22 X-Ray spectra

23 Molecules - Polar Bond

24 Heitler-London theory of H_2- molecule

25 Single orbit approximation - ~~Wang~~ Ritz method - Wang eigenfunctions

26 Theory of Van der Waals forces

27 Terms of a diatomic molecule Rotation-oscillation bands

28 Electronic band spectra

29 Alternating intensities

30 Formula for prob. of transition $\frac{2\pi}{h} |\mathcal{H}_{nm}|^2 N$

31 Born approximation of collision theory

32 Case of spherical symmetry - Case of identical particles

33 The exact theory of collision for ~~against~~ ~~[illegible]~~ central ~~spherical~~ forces, ~~center~~

ROYAL V

VERNON

COMPOSITION BOOKS

NO.	LVS.	SIZE
101-40	40	$9\frac{3}{4} \times 7\frac{5}{8}$
$101\frac{1}{4}$	48	$9\frac{3}{4} \times 7\frac{5}{8}$
101	60	$9\frac{3}{4} \times 7\frac{5}{8}$
101X	72	$9\frac{3}{4} \times 7\frac{5}{8}$
102	84	$9\frac{3}{4} \times 7\frac{5}{8}$
103	96	$9\frac{3}{4} \times 7\frac{5}{8}$
104	120	$9\frac{3}{4} \times 7\frac{5}{8}$
$104\frac{1}{2}$	144	$9\frac{1}{4} \times 7\frac{5}{8}$