Conjugate Systems
With Indeterminate Axis Curves.

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BY ERNEST P. LANE.

I. INTRODUCTION.

The general projective theory of congruences may be studied in connection with a completely integrable system of partial differential equations of the form

\[ \begin{align*}
  y_v &= mz, \quad z_u = ny, \\
  y_{uu} &= a y + b z + c y_u + d z_v, \\
  z_{vv} &= a'y + b'z + c'y_u + d'z_v,
\end{align*} \]

(D)

where the subscripts indicate differentiations. In order that system (D) may be completely integrable, its coefficients, which are assumed to be analytic functions of the two arguments \( u \) and \( v \), must satisfy the following integrability conditions:

\[ \begin{align*}
  c &= f_u, \quad d' = f_v, \\
  b &= -d_u + df_v, \quad a' = -c_u' + c'f_u, \\
  mn - c'd &= f_{uv}, \\
  m_{uu} + d_v + df_{vv} + d_{uv} - f_{um} &= ma + db', \\
  n_{vv} + c_{uu} + c'f_{uu} + c'u + f_{nu} - f_{vm} &= nb' + c'a, \\
  2m_{uv} + mn_u &= a_v + f_u mn + a'd, \\
  m_o n + 2mn_o &= b'_u + f_v mn + b'o,
\end{align*} \]

where \( f \) is an analytic function of \( u \) and \( v \). Then system (D) has exactly four pairs of linearly independent solutions, \( (y^{(i)}, z^{(i)}), (i = 1, \ldots, 4) \), such that the general solution is of the form

\[ \begin{align*}
  y &= \sum_{i=1}^{4} c^{(i)} y^{(i)}, \\
  z &= \sum_{i=1}^{4} c^{(i)} z^{(i)},
\end{align*} \]

where \( c^{(1)}, \ldots, c^{(4)} \) are arbitrary constants. If \( y^{(1)}, \ldots, y^{(4)} \) and \( z^{(1)}, \ldots, z^{(4)} \) are interpreted as the homogeneous coördinates of two points \( P_y \) and \( P_z \) of three-dimensional space, these points will describe in general, for variable \( u \) and \( v \), two surfaces \( S_y \) and \( S_z \), but the possibility is not excluded that these surfaces reduce to curves. The line \( P_y P_z \) describes a congruence whose

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focal surfaces are \( S_y \) and \( S_z \), and whose developables are obtained by equating either \( u \) or \( v \) to a constant.

The form of system \((D)\) is undisturbed by the group of transformations
\[
(T) \quad y = \lambda(u)\tilde{y}, \quad z = \mu(v)\bar{z}, \quad \tilde{u} = \alpha(u), \quad \bar{v} = \beta(v),
\]
where \( \lambda, \mu, \alpha, \beta \) are arbitrary functions of the arguments indicated. Among the invariants of this group are the four coefficients \( m, n, c', d \). It is easy to verify that \( I \) and \( l \) are also invariants, where
\[
I = a + c \frac{m_u}{m} - \frac{m_{uu}}{m}, \quad l = b \frac{n_v}{d} + \frac{n_v}{n}.
\]

Five fundamental covariants are \( y, z, \rho, \sigma, \epsilon \), where
\[
\rho = y_u - \frac{m_u}{m} y, \quad \sigma = z_v - \frac{n_v}{n} z, \quad \epsilon = \rho_u - \frac{m_{1u}}{m_1} \rho,
\]
and
\[
m_1 = m \left( mn - \frac{\partial^2}{\partial u \partial v} \log m \right).
\]

The axis of a point \( P \) on a surface has been defined,* with reference to a given conjugate system of curves on the surface, to be the line of intersection of the osculating planes at \( P \) of the two curves of the system which pass through \( P \). The axis congruence of a surface is composed of the axes of all its points. An axis curve of a surface is a curve such that the axes of its points form a developable of the axis congruence.

The curves \( u = \text{const.} \) and \( v = \text{const.} \) form a conjugate system on \( S_y \). For convenience of reference we quote here some results,† concerning the axis curves of \( S_y \) defined with reference to this conjugate system. The axis curves are given by the equation
\[
(1) \quad d_1\delta u^2 - mL\delta u\delta v - c'dm\delta v^2 = 0,
\]
where
\[
d_1 = d \left( c'd - \frac{\partial^2}{\partial u \partial v} \log d \right).
\]

If the function \( \tau \) be defined by
\[
(2) \quad \tau = lx + \sigma,
\]
then the point \( P_\tau \) is the point of intersection of the axis of \( P_y \) with the line \( P_xP_\tau \). The foci of the axis are determined by factoring the covariant
\[
(3) \quad c'dy^2 - mL\tau - d\tau^2.
\]


† Congruences, § 2.
It has been remarked* that the axis curves are indeterminate if

\[ d_1 = mI = c'dm = 0. \]

It is the purpose of this paper to pursue the consequences of this remark, and to study those surfaces upon which there exists a conjugate system with indeterminate axis curves.

The author takes advantage of this opportunity to express his appreciation of the many helpful suggestions so kindly given him by Professor Wilczynski while this paper was in process of preparation.

I. Fundamental Theorem.

We shall now assume that the axis curves on \( S_y \) defined with reference to the parametric conjugate system, are indeterminate. Then the equations

\[ d_1 = mI = c'dm = 0 \]

are valid. If the surface \( S_y \) does not reduce to a curve, the invariant \( m \) does not vanish.† Similarly, if \( S_z \) is not degenerate, \( n \neq 0 \). Moreover, if \( S_y \) is not a developable surface, \( d \neq 0 \). We shall assume from now on that we are considering only those surfaces \( S_y \) which are nondegenerate and nondevelopable, and surfaces \( S_z \) which at least are nondegenerate. Under these assumptions we shall proceed to find formulas for the coefficients of system \((D)\).

Equations (3) may be replaced by the equivalent equations

\[ \frac{\partial^2}{\partial u \partial v} \log d = I = c' = 0. \]

As an immediate consequence we have

\[ d = UV, \]

where \( U \) and \( V \) are nonvanishing functions of \( u \) alone and of \( v \) alone, respectively.

When use is made of equations (4) and (5), the integrability conditions may be written in the form

\[ c = f_u, \quad d' = f_v, \quad b = -UV' - UVf_v, \quad a' = 0, \]

\[ mn = f_{uv}, \quad V'' + Vf_{uv} + V'f_v = Vb', \quad n_u - f_vn_v = nb', \]

\[ 2m_u n + mn_u = a_v + f_u mn, \]

\[ m_v n + 2mn_v = b'_v + f_v mn, \]

* Congruences, p. 317.
† Brussels Paper, p. 28.
where $V'$, $V''$, etc., denote derivatives. On eliminating $b'$ between the sixth and ninth conditions, on making use of the fifth condition, and on integrating with respect to $v$ there results

$$n = U_1 V e',$$

(6)

where $U_1$ is an arbitrary nonvanishing function of $u$ alone. By means of (6) and the fifth and eighth conditions we find, after integrating with respect to $v$,

$$a = 2f_{uu} - f_u \frac{U_1'}{U_1} - f_u^2 - U_2,$$

where $U_2$ is still another arbitrary function of $u$ only.

Let us now regard the functions $f$, $V$, $U$, $U_1$, $U_2$ as given functions, and let us consider the possibility of defining the coefficients of a system $(D)$ by the following formulas:

$$m = \frac{f_{uw}e'}{U_1 V},$$

$$n = U_1 V e'.$$

$$a = 2f_{uu} - f_u \frac{U_1'}{U_1} - f_u^2 - U_2,$$

$$b = -UV' - UVf_v,$$

$$c = f_u,$$

$$d = UV,$$

$$a' = 0,$$

$$b' = \frac{V''}{V} + f_{vv} + \frac{V'}{V} f_v,$$

$$c' = 0,$$

$$d' = f_v.$$

(7)

When the coefficients thus defined are substituted in the integrability conditions, all of them are found to be satisfied identically except the sixth. This condition is found to be equivalent to $I = 0$. When it is written out in terms of the given functions, it becomes

$$\frac{f_{uuw}}{f_{vw}} - 3f_u \frac{f_{uw}}{f_{uw}} - 3f_{uu} + 3f_u^2 - \frac{2U_1' f_{uvw}}{U_1 f_{uv}}$$

$$+ 4f_u \frac{U_1'}{U_1} + 2 \left( \frac{U_1'}{U_1} \right)^2 \frac{U_1''}{U_1} + U_2 = 0.$$  

(8)

Therefore if the functions $f$, $U_1$, $U_2$ satisfy equation (8), formulas (7) furnish the coefficients of a system $(D)$ which satisfies all of the requirements.

Our results may be summarized in the following fundamental theorem. Let $U$, $U_1$, and $U_2$ be arbitrary functions of $u$ alone such that $UU_1 \neq 0$. Furthermore let $f$ be a function of $u$ and $v$ satisfying equation (8) and such that $f_{uw} \neq 0$. Finally let $V$ be an arbitrary nonvanishing function of $v$ alone. Then formulas (7) furnish the coefficients of the most general system of the form
(D) for which \( S_y \) is nondegenerate and nondevelopable and \( S_x \) nondegenerate, and for which the axis curves on \( S_y \), defined with reference to the parametric conjugate system, are indeterminate. The integrability conditions for such a system (D) are all satisfied identically.

III. A SIMPLIFYING TRANSFORMATION.

All transformations of the group (T) are without geometrical significance; we shall therefore employ this group to simplify our analysis. The effect of (T) on (D) is to transform (D) into another system of the same form, of which the coefficients have the following values.*

\[
\begin{align*}
m &= \frac{\mu}{\lambda} \frac{1}{\beta_v} m, & n &= \frac{\lambda}{\mu} \frac{1}{\alpha_u} n, \\
\tilde{a} &= \frac{1}{\alpha_u} \left( a + \frac{\lambda_u}{\lambda} c - \frac{\lambda_{uv}}{\lambda} \right), & \tilde{b}' &= \frac{1}{\beta_v} \left( b' + \frac{\mu_u}{\mu} d' - \frac{\mu_{uv}}{\mu} \right), \\
\tilde{b} &= \frac{1}{\alpha_u} \left( b + \frac{\mu_u}{\mu} d \right), & \tilde{a}' &= \frac{1}{\beta_v} \left( a' + \frac{\lambda_u}{\lambda} c' \right), \\
\tilde{c} &= \frac{1}{\alpha_u} \left( c - 2 \frac{\lambda_u}{\lambda} - \frac{\alpha_{uu}}{\alpha_u} \right), & \tilde{d}' &= \frac{1}{\beta_v} \left( d' - 2 \frac{\mu_u}{\mu} - \frac{\beta_{uv}}{\beta_v} \right), \\
\tilde{d} &= \frac{\beta_v}{\alpha_u} \frac{\mu}{\lambda} d, & \tilde{c}' &= \frac{\alpha_u}{\beta_v} \frac{\lambda}{\mu} c'.
\end{align*}
\]

In the first place we observe that the product \( UV \) is reduced to unity by a transformation of the group (T) for which

\[
\lambda = U, \quad \mu V = \alpha_u = \beta_v = 1.
\]

When this transformation has been made the coefficient \( \tilde{d} \) of the transformed system becomes equal to unity, and the integrability conditions reduce to

\[
\begin{align*}
c &= f_u, & d' &= f_v, & b &= -f_v, & a' &= 0, \\
mn &= f_{uv}, & b' &= f_{vv}, & n_{vv} &= f_v n_v + nb', & \\
2mn_n + mn_u &= a_v + mnf_u, \\
m_v n + 2mn_v &= b'_v + mnf_v.
\end{align*}
\]

On eliminating \( b' \) between the sixth and ninth conditions, on making use of the fifth condition, and on integrating with respect to \( v \), there results

\[
n = U x e^f.
\]

The largest subgroup of the group (T) which preserves the condition \( \tilde{d} = 1 \) is given by the relations

\[
\alpha_u^2 \lambda = \beta_v \mu = \text{const.},
\]

* Brussels Paper, Equations (16) and (22).
where \( \lambda \) and \( \mu \) are still arbitrary functions. The effect of this subgroup on the coefficient \( n \) is given by

\[
\bar{n} = \frac{\beta_v}{\alpha_v} n = \frac{\beta_v}{\alpha_v} U_1 e'.
\]

Therefore the function \( U_1 \) is reduced to unity by a transformation for which

\[
\alpha_v^2 \lambda = \mu = 1, \quad \alpha_v^3 = U_1, \quad \beta_v = 1.
\]

When this transformation has been made, the fifth integrability condition gives

\[
m = f_{uv} e^{-f},
\]

and then the relation \( I = 0 \) becomes

\[
a = \frac{f_{uuuv}}{f_{uv}} - \frac{3f_u f_{uv}}{f_{uv}} - f_{uu} + 2f_u^2.
\]

Now regarding \( f \) as a given function, we consider the possibility of defining the coefficients of a system \( (D) \) by the following formulas:

\[
m = f_{uv} e^{-f}, \quad n = e',
\]

\[
a = \frac{f_{uuuv}}{f_{uv}} - \frac{3f_u f_{uv}}{f_{uv}} - f_{uu} + 2f_u^2 \quad b = - f_v.
\]

\[
c = f_v, \quad d = \frac{c}{u},
\]

\[
a' = 0, \quad b' = \frac{c}{uv},
\]

\[
c' = 0, \quad d' = f_v.
\]

When the coefficients thus defined are substituted in the integrability conditions, all of them are found to be satisfied identically except the eighth, which becomes

\[
f_{uuuv} - \frac{3f_u f_{uv}}{f_{uv}} - \frac{f_{uuuv} f_{uv}}{(f_{uv})^2} + \frac{3f_u f_{uv} f_{uv}}{(f_{uv})^2} - 6f_{uuu} + 6f_u f_{uv} = 0.
\]

Therefore if \( f \) satisfies equation (12), then formulas (11) furnish the coefficients of a system \( (D) \) such that the axis curves of the parametric conjugate system are indeterminate.

The apparently formidable equation (12) can be very much simplified. On rearranging the terms and integrating with respect to \( v \), we obtain

\[
f_{uuuv} - \frac{3f_u f_{uv}}{f_{uv}} - f_{uu} + 3f_v^2 + U_2 = 0,
\]

where \( U_2 \) is an arbitrary function of \( u \) alone. If equation (13) be multiplied through by \( f_{uv} \), it can be integrated with respect to \( v \), giving

\[
f_{uuu} - 3f_u f_{uu} + f_v^2 + U_2 f_u = U_3,
\]
where $U_3$ is another arbitrary function of $u$ alone. Finally we introduce the function $F$ by the definition
\begin{equation}
(15) \quad f = - \log F,
\end{equation}
whereupon (14) reduces to the linear homogeneous equation
\begin{equation}
(16) \quad \frac{\partial^3 F}{\partial u^3} + U_2 \frac{\partial F}{\partial u} + U_3 F = 0.
\end{equation}
We remark that as a consequence of (15) we have
\begin{equation}
F^2f_{uv} + FF_{uv} - F_uF_v = 0.
\end{equation}

We may now state the following theorem. Let $U_2$ and $U_3$ be arbitrary functions of the single variable $u$. Let $F$ be a function of $u$ and $v$ which is a solution of (16) but which is not a solution of $FF_{uv} - F_uF_v = 0$. Let the function $f$ be determined from $f = - \log F$. Then the most general system $(D)$ which is of interest in the theory of indeterminate axis curves can be reduced, by means of transformations of group (T), to the system:
\begin{equation}
(\Delta) \quad \begin{align*}
y_u &= f_{uv}e^{-f_v}, \quad z_u = e^f y, \\
y_{uu} &= \left( f_{uuu} - \frac{3f_{uf_{uv}}}{f_{uv}} - f_{uu} + 2f_u^2 \right) y - f_{uv}z + f_{uy} + z_v, \\
z_{uv} &= f_{vz} + f_vz_v.
\end{align*}
\end{equation}
The integrability conditions for system $(\Delta)$ are all satisfied identically. We shall base our further theory on this system.

IV. THE FIRST LAPLACE TRANSFORMED CONGRUENCE.

In the theory of indeterminate axis curves, the first Laplace transformed congruence, whose focal surfaces are $S_\rho$ and $S_\nu$, and the original congruence, whose focal surfaces are $S_u$ and $S_\nu$ and whose equations are system $(D)$, are equally important. We shall need to set up the system $(D_1)$ of equations of the first Laplace transformed congruence. But before doing so let us define a new function $g$ by the equation
\begin{equation}
(17) \quad g = 2f - \log f_{uv}.
\end{equation}
As thus defined, $g$ plays the same rôle in the theory of the first Laplace transformed congruence that $f$ plays in the theory of the original congruence.

By means of equation (17) it is possible to write (13) in the symmetrical form
\begin{equation}
(18) \quad f_u^2 + g_u^2 - f_{uu} - g_{uu} - f_u g_u + U_2 = 0.
\end{equation}
Differentiation of (18) with respect to $v$ yields
\begin{equation}
(19) \quad 2f_{uf_v} + 2g_{ug_v} - f_{uvu} - g_{uvu} - f_{ug_v} + f_u g_v = 0.
\end{equation}
Now the sum of the first, third, and fifth terms of this equation is easily shown to be zero, by using equation (17). Therefore, if we suppose \( g_{uv} \neq 0 \); an assumption whose geometrical meaning will be made clear later, we have

\[
(f_u = 2g_u - \frac{g_{uvv}}{g_{uv}}).
\]

We obtain, on integrating with respect to \( u \),

\[
f = 2g - \log g_{uv} + \log V,
\]

where \( V \) is an arbitrary function of \( v \) alone.

We may make a transformation to reduce \( V \) to unity. It is readily verified that all transformations of the group \( (T) \) for which

\[
\lambda = \alpha_u, \quad \alpha^*_v = \beta_v \mu = 1,
\]

leave the form of system \( (\Delta) \) unchanged, although \( f \) is thereby transformed according to the formula

\[
\tilde{f} = f - \log \mu.
\]

Moreover, \( g \) is transformed according to the formula

\[
\tilde{g} = g - 3 \log \mu,
\]

as is seen upon using equation (17). If then we choose

\[
\lambda = 1, \quad \mu = V^{-\frac{1}{3}}, \quad \alpha_u = 1, \quad \beta_v = V^\frac{1}{3},
\]

the conditions (22) are satisfied and we find, on making the necessary calculations, that

\[
\tilde{f} = 2\tilde{g} - \log \tilde{g}_{uv},
\]

so that the transformed \( \tilde{V} \) is unity. Equation (17) may now be solved for \( f \) in the form

\[
f = 2g - \log g_{uv}.
\]

When \( f \) is eliminated from equation (14), the result is

\[
g_{uuu} - 3g_u g_{uu} + g_u^2 + U_2 g_u - U_2' + U_3 = 0.
\]

Let us introduce the function \( G \) by the definition

\[
g = -\log G.
\]

Then (27) is equivalent to

\[
\frac{\partial^3 G}{\partial u^3} + U_2 \frac{\partial G}{\partial u} + (U_2' - U_2) G = 0.
\]

It will be observed that this equation is the Lagrange adjoint of equation (16), which is satisfied by \( F \), the derivatives in both equations being taken with respect to the variable \( u \) only.
It is interesting at this point to see that \( F \) and \( G \) satisfy also certain equations in which differentiations occur only with respect to the variable \( v \). The truth of the equation

\[
2f_{y}f_{uv} + 2g_{v}g_{uv} - f_{vv} - g_{vuv} - f_{uv}g_{v} - g_{vn}f_{v} = 0
\]

is established by observing that alternate terms taken together form two sums, one of which vanishes in virtue of equation (17), and the other in virtue of (26). Upon integrating with respect to \( u \), we obtain

\[
f_{v}^{2} + g_{v}^{2} - f_{vv} - g_{v} - f_{v}g_{v} + V_{2} = 0,
\]

where \( V_{2} \) is an arbitrary function of \( v \) alone. When \( f \) is eliminated from this equation, there results

\[
\frac{g_{vuv}}{g_{uv}} - \frac{3g_{v}g_{vuv}}{g_{uv}} - 3g_{vv} + 3g_{v}^{2} + V_{2} = 0.
\]

Upon integrating with respect to \( u \) again, we obtain

\[(30) \quad g_{vuv} - 3g_{v}g_{vv} + g_{v}^{3} + V_{2}g_{v} = V_{3},\]

where \( V_{3} \) is a function of \( v \) only. Equation (30) is equivalent to

\[(31) \quad \frac{\partial^{3}G}{\partial v^{3}} + V_{2} \frac{\partial G}{\partial v} + V_{3}G = 0.\]

Similarly, if \( g \) is eliminated from (30), there results

\[
f_{vv} - 3f_{v}f_{vv} + f_{v}^{3} + V_{2}f_{v} - V_{2}^{2} + V_{3} = 0,
\]

and this equation is equivalent to

\[(32) \quad \frac{\partial^{3}F}{\partial v^{3}} + V_{2} \frac{\partial F}{\partial v} + (V_{2}^{2} - V_{3})F = 0.\]

Equation (32) is the Lagrange adjoint of (31).

Making use of the function \( g \), let us place*

\[(33) \quad \rho = y_{u} - (f_{u} - g_{u})y, \quad \xi = e^{\rho - v}y,
\]

and find the system \((\Delta_{1})\) of equations of the first Laplace transformed congruence. On differentiating (33) and employing system \((\Delta)\), we find that \( \rho \) and \( \xi \) satisfy

\[
\rho_{v} = e^{\xi} \xi, \quad \xi_{u} = e^{\rho - v} \rho,
\]

\[(\Delta_{1}) \quad \rho_{uu} = g_{uu} \rho + g_{u} \rho_{u},\]

\[
\xi_{vv} = - g_{u} \rho + (g_{vv} - f_{uv} - f_{v}g_{v} + f_{v}^{2}) \xi + \rho_{u} + g_{v} \xi_{v}.
\]

V. Geometrical Theorems.

The three covariants \( \rho, \sigma, \epsilon \) for system \((\Delta)\), when written in terms of \(f\) and \(g\), are

\[
(34) \quad \rho = y_u - (f_u - g_u)y, \quad \sigma = z_v - f_vz, \quad \epsilon = \rho_u - g_u\rho.
\]

These equations, together with systems \((\Delta)\) and \((\Delta_1)\) will now be used to deduce a number of geometric propositions.

From the first of equations (34) we find, by differentiation with respect to \(u\) and use of system \((\Delta)\), that \(\epsilon = \sigma\). Moreover we find \(\sigma_u = \sigma_v = 0\).

Therefore, if the axis curves of a conjugate net are indeterminate, the second and minus second Laplace transforms of the net are identical and reduce to a single fixed point.

The third equation of system \((\Delta_1)\) shows that the lines \(v = \text{const.}\) on \(S_v\) are straight lines, while the third of equations (34) shows that these lines all pass through \(P_v\). Therefore the surface \(S_v\) is a cone with its vertex at \(P_v\). In the same way we conclude from the fourth equation of system \((\Delta)\) that the lines \(u = \text{const.}\) on \(S_u\) are straight lines, and from the second of equations (34) that these lines all pass through \(P_u\). Therefore, if the axis curves of a conjugate net are indeterminate, the first and minus first Laplace transforms of the net are on two cones with a common vertex at the fixed point into which the second and minus second Laplace transforms degenerate.

Inspection of the covariant \((C)\), namely

\[
(C \text{ bis}) \quad c'd'y^2 - mlyr - dm'T^2,
\]

shows that, when the axis curves on \(S_v\) are indeterminate, the foci of the axis of \(P_v\) coincide at the point \(P_v\). Moreover equation (2) shows that \(P_v\) coincides with the point \(P_v\). Therefore if the axis curves of a conjugate system on a surface are indeterminate, the axis congruence reduces to a bundle of lines with its vertex at the fixed point into which the second and minus second Laplace transforms of the surface degenerate.

Let us differentiate twice with respect to \(v\) the first equation of system \((\Delta)\). From the equations thus obtained let us eliminate \(z, z_v, \) and \(z_{vv},\) using the fourth equation of system \((\Delta)\). There results

\[
(35) \quad y_{vvv} + (2g_v - 3f_v)y_{vv} + (g_{vv} + g^2 + 2f_v^2 - 2f_{vv} - 3f_vg_v)y_v = 0,
\]

the equation of the curves \(u = \text{const.}\) on \(S_y\). Since this equation is of the third order we conclude that the curves \(u = \text{const.}\) are plane curves. Similarly we differentiate once with respect to \(u\) the third equation of system \((\Delta)\) and eliminate \(z, z_v,\) and \(z_{uv}\) to obtain

\[
(36) \quad y_{uuu} - f_u y_{uu} + (g_{uu} - 2f_{uu} + f_u g_u - g_u^2)y_u + (g_{uu} - f_{uu} + f_u g_u) y = 0,
\]
the equation of the curves \( v = \text{const.} \) on \( S_v \). These curves are also plane curves. In this way we see that, if the axis curves of a conjugate system are indeterminate, this conjugate system consists of two one-parameter families of plane curves.

VI. INTEGRATION OF SYSTEM \((\Delta)\).

It has been shown that, for system \((\Delta)\), the two surfaces \( S_\sigma \) and \( S_\iota \) reduce to the same fixed point, which may be denoted indifferently by \( P_\sigma \) or \( P_\iota \). Since the fundamental tetrahedron of reference is arbitrary, let us choose it so that \( P_\sigma \) may have \((0, 0, 0, 1)\) for its coördinates. Then the second of equations (34) furnishes

\[
\begin{align*}
\varphi^{(k)} - f_z \varphi^{(k)} &= 0, \quad (k = 1, 2, 3), \\
\varphi^{(4)} - f_z \varphi^{(4)} &= 1.
\end{align*}
\]

After integrating with respect to \( v \), we obtain

\[
\begin{align*}
z^{(k)} &= \varphi^{(k)} e^f, \\
z^{(4)} &= \varphi^{(4)} e^f + e^f \int e^{-f} dv,
\end{align*}
\]

where the four functions \( \varphi \) are functions of \( u \) alone, and are as yet arbitrary. By the indicated quadrature, we mean the definite integral from a fixed lower limit to a variable upper limit, \( u \) being regarded as fixed.

We now use the second equation of system \((\Delta)\) to determine four values of \( y \), and obtain

\[
\begin{align*}
y^{(k)} &= \varphi^{(k)} f_u + \varphi^{(k)} u, \\
y^{(4)} &= \varphi^{(4)} f_u + \varphi^{(4)} u + f_u \int e^{-f} dv - \int e^{-f} f_u dv.
\end{align*}
\]

When corresponding values of \( y \) and \( z \) are substituted in the first and fourth equations of system \((\Delta)\), these equations are found to be satisfied identically without restriction on the functions \( \varphi \). But when corresponding values of \( y \) and \( z \) are substituted in the third equation, it is found that the functions \( \varphi \) must be solutions of certain ordinary differential equations. In fact \((y^{(k)}, z^{(k)}), \ (k = 1, 2, 3)\), constitute three pairs of linearly independent solutions of system \((\Delta)\) if, and only if, the \( \varphi^{(k)} \) are three linearly independent solutions of the equation*

\[
\frac{d^3 \varphi}{du^3} + U_2 \frac{d \varphi}{du} + U_3 \varphi = 0,
\]

* Since the second derivative is missing, the Wronskian of any three linearly independent solutions is a constant, which may be supposed to be unity.
and \((y^{(4)}, z^{(4)})\) are a pair of solutions of system \((\Delta)\) if, and only if, \(\varphi^{(4)}\) is a solution of the equation

\[
\frac{d^3 \varphi}{du^3} + U_2 \frac{d\varphi}{du} + U_3 \varphi = 1.
\]

Equations (37) and (16) are to be compared.

Let us denote the three second order Wronskians of the \(\varphi^{(k)}\) by \(U^{(k)}\), so that, for instance,

\[
U^{(1)} = \varphi_u^{(2)} \varphi^{(3)} - \varphi_u^{(3)} \varphi^{(2)}.
\]

Then the functions \(U^{(k)}\) are solutions of the Lagrange adjoint of (37), namely

\[
\frac{d^3 U}{du^3} + U_2 \frac{dU}{du} + (U_2' - U_3) U = 0.
\]

This equation is to be compared with equation (29) which is satisfied by \(G\).

We remark further that comparison of (37) and (38) shows that \(\varphi^{(4)}\) can be expressed in terms of the \(\varphi^{(k)}\) by the well-known method of variation of parameters. We find in this way

\[
\varphi^{(4)} = \sum_{k=1}^{3} \left( \int U^{(k)} du + c^{(k)} \right) \varphi^{(k)},
\]

where the \(c^{(k)}\) are arbitrary constants, and the indicated quadratures denote again definite integrals from a common fixed lower limit to a variable upper limit.

In the process of integrating system \((\Delta)\) we have used the second of equations (34), but we might just as well have used the third. This equation gives

\[
\rho_u^{(k)} - g_u \rho^{(k)} = 0, \quad (k = 1, 2, 3),
\]

\[
\rho_u^{(4)} - g_u \rho^{(4)} = 1.
\]

On integrating with respect to \(u\) we obtain

\[
\rho^{(k)} = \varphi^{(k)} e^\sigma,
\]

\[
\rho^{(4)} = \varphi^{(4)} e^\sigma + e^\sigma \int e^{-\sigma} du,
\]

where the functions \(\varphi\) are functions of \(v\) alone, and are as yet arbitrary. The first equation of system \((\Delta_1)\) gives

\[
\xi^{(k)} = \varphi^{(k)} + g_v \varphi^{(k)},
\]

\[
\xi^{(4)} = \varphi^{(4)} + g_v \varphi^{(4)} + g_v \int e^{-\sigma} du - \int e^{-\sigma} g_v du.
\]
Then the second of equations (33) furnishes four values for \( y \), and the first of system (\( \Delta \)) furnishes four values for \( z \).

We find that the \( \psi^{(k)} \) are solutions of

\[
\frac{d^3\psi}{dv^3} + V_2 \frac{d\psi}{dv} + V_3 \psi = 0,
\]

and \( \psi^{(4)} \) is a solution of

\[
\frac{d^3\psi}{dv^3} + V_2 \frac{d\psi}{dv} + V_3 \psi = 1.
\]

We denote the three Wronskians of the \( \psi^{(k)} \) by \( V^{(k)} \), so that, for instance,

\[
V^{(1)} = \psi^{(2)} \psi^{(3)} - \psi^{(3)} \psi^{(2)}.
\]

Then by variation of parameters we find

\[
\psi^{(4)} = \sum_{k=1}^{3} \left( \int V^{(k)} dv + c^{(k)} \right) \psi^{(k)}.
\]

Equation (41) should be compared with equation (31) which is satisfied by \( G \), while the \( V^{(k)} \) satisfy the Lagrange adjoint of (41), namely

\[
\frac{d^3V}{dv^3} + V_2 \frac{dV}{dv} + (V'_2 - V_3) V = 0.
\]

This equation is to be compared with (32), which is satisfied by \( F \).

We are able to express \( F \) and \( G \) in terms of the functions \( \varphi^{(k)} \) and \( \psi^{(k)} \). In fact, \( F \) is a solution of (16) and (32), while \( G \) is a solution of (29) and (31). Now, reference to (37), (39), (41), and (44) shows that we have

\[
F = \sum_{k=1}^{3} V^{(k)} \varphi^{(k)}, \quad G = \sum_{k=1}^{3} U^{(k)} \psi^{(k)},
\]

where the \( V^{(k)} \) and \( U^{(k)} \) are the Wronskians previously defined.

Recalling then that \( f = - \log F, g = - \log G \) we can show that the values obtained for \( y \) by both methods of integrating system (\( \Delta \)) reduce to the following:

\[
Fy^{(1)} = U^{(3)} V^{(2)} - U^{(2)} V^{(3)},
\]

\[
Fy^{(2)} = U^{(1)} V^{(3)} - U^{(3)} V^{(1)},
\]

\[
Fy^{(3)} = U^{(2)} V^{(1)} - U^{(1)} V^{(2)},
\]

\[
y^{(4)} = \sum_{k=1}^{3} \left( \int U^{(k)} du + \int V^{(k)} dv + c^{(k)} \right) y^{(k)}.
\]
The values for \( z \), obtained by both methods, reduce to

\[
F_2^{(k)} = \varphi^{(k)}, \quad (k = 1, 2, 3),
\]

\[
1z^{(4)} = \sum_{k=1}^{3} \left( \int U^{(k)} du + \int V^{(k)} dv + c^{(k)} \right) z^{(k)},
\]

while the values for \( \rho \) reduce to

\[
G\rho^{(k)} = \psi^{(k)}, \quad (k = 1, 2, 3),
\]

\[
1\rho^{(4)} = \sum_{k=1}^{3} \left( \int U^{(k)} du + \int V^{(k)} dv + c^{(k)} \right) \rho^{(k)}.
\]

VII. Geometrical Applications.

We shall now proceed to apply formulas (46), which were obtained by integrating system (\( \Delta \)). These formulas give the coördinates of an arbitrary point \( P_y \) on the most general surface \( S_y \) which has the property that the axis curves of the parametric conjugate net are indeterminate.

Let us first deduce the equation of the plane tangent to \( S_y \) at \( P_y \). This plane is determined by the three points \( y, y_u, \) and \( y_v, \) and when its equation is found in the ordinary way, it turns out to be

\[
x_4 = \sum_{k=1}^{3} \left( \int U^{(k)} du + \int V^{(k)} dv + c^{(k)} x_k \right).
\]

This equation has the same form as the last of (46). As is well known, this plane osculates at \( P_\rho \) the curve \( u = \text{const.} \) through \( P_\rho \) on the cone \( S_\rho \), and also osculates at \( P_z \) the curve \( v = \text{const.} \) through \( P_z \) on the cone \( S_z \). The two families of curves, \( u = \text{const.} \) on \( S_\rho \), and \( v = \text{const.} \) on \( S_z \), may then be said to have the same set of osculating planes, namely the set of tangent planes of \( S_y \).

If we differentiate equation (47) with respect to \( u \), we obtain

\[
\sum_{k=1}^{3} U^{(k)} x_k = 0.
\]

This plane cuts the surface \( S_y \) in the curve \( u = \text{const.} \) which passes through \( P_y \), and moreover is tangent to the cone \( S_z \) along the corresponding generator \( u = \text{const.} \). In the same way, we obtain by differentiating equation (47) with respect to \( v \),

\[
\sum_{k=1}^{3} V^{(k)} x_k = 0.
\]

This plane cuts \( S_y \) in the curve \( v = \text{const.} \) which passes through \( P_y \), and is tangent to the cone \( S_\rho \) along a generator \( v = \text{const.} \).
The functions $U^{(k)}$ are the coördinates of the enveloping plane of the cone $S_z$, and the $V^{(k)}$ are the coördinates of the enveloping plane of $S_p$. Reference to equations (46) shows that the coördinates of $P_y$ depend only on these six functions and the three arbitrary constants $c^{(k)}$. So then, except for these three constants, the surface $S_y$ is determined when the cones $S_z$ and $S_p$ are given.

Let us seek for a geometrical meaning for the constants $c^{(k)}$. To this end, let us consider the following projective transformation:

$$\begin{align*}
\omega x'_1 &= x_1, \\
\omega x'_2 &= x_2, \\
\omega x'_3 &= x_3, \\
\omega x'_4 &= a^{(1)} x_1 + a^{(2)} x_2 + a^{(3)} x_3 + x_4.
\end{align*}$$

(50)

This transformation is completely characterized by the property of leaving invariant the point $(0, 0, 0, 1)$ and every stright line through this point. It therefore leaves $P_x$ and the cones $S_z$ and $S_p$ invariant. But it changes the surface $S_y$, so that the first three coördinates of $P_y$ remain the same as given by (46), while the fourth becomes

$$y^{(k)} = \sum_{k=1}^{3} \left( \int U^{(k)} du + \int V^{(k)} dv + c^{(k)} + a^{(k)} \right) y^{(k)}.$$

Therefore assigning various values to the constants $c^{(k)}$ merely amounts to making projective transformations of the simple type (50). In other words, the surface $S_y$ is determined by the cones $S_z$ and $S_p$ up to a projective transformation of this type.

We may now summarize our results. Let there be given two cones with a common vertex. Then there exists a three-parameter family of surfaces, each surface having the following properties:

1. The tangent planes of the two cones cut the surface in a conjugate system.
2. The axis curves of this system are indeterminate.
3. The axis of every point on the surface, defined with reference to this system, passes through the common vertex of the given cones. Moreover, the family of surfaces has the property that any surface of it may be obtained from any other surface of it by means of the projective transformation that leaves invariant the common vertex of the two given cones and also every line through this point.

VIII. A Special Case.

We have assumed throughout this work that $g_{uv} f_{uv} \neq 0$. The fifth integrability condition, when $c' = 0$ and $S_y$ is nondegenerate, shows that
\( f_{uv} = 0 \) if, and only if, \( S_z \) degenerates into a curve. This curve would be a straight line, since \( S_z \) is also a developable. By analogy, \( g_{uv} = 0 \) if, and only if, \( S_z \) reduces to a straight line. Now it happens here, as it often does happen, that the essential results of the investigation are true independently of certain restricting hypotheses used in deriving them. For example, equations (46) are valid, even if one or both of the two cones do reduce to straight lines.

Let us now specialize the six functions and three constants appearing in (46) so as to obtain a very simple surface which has on it a conjugate system with indeterminate axis curves. In the last of equations (46) we shall take the lower limit of integration to be zero both for \( u \) and for \( v \). Then we shall take

\[
\begin{align*}
U^{(1)} &= -1, & U^{(2)} &= 0, & U^{(3)} &= u, \\
V^{(1)} &= 0, & V^{(2)} &= -1, & V^{(3)} &= v, \\
c^{(1)} &= 0, & c^{(2)} &= 0, & c^{(3)} &= -\frac{1}{2}.
\end{align*}
\]

In this way equations (46) reduce to

\[
(51) \quad F_y^{(1)} = -u, \quad F_y^{(2)} = -v, \quad F_y^{(4)} = -1, \\
F_y^{(4)} = \frac{1}{2}(u^2 + v^2 + 1).
\]

Let us introduce nonhomogeneous coordinates by the equations

\[
x = y^{(1)}/y^{(4)}, \quad y = y^{(3)}/y^{(4)}, \quad z = y^{(3)}/y^{(4)}.
\]

Then from (51) we obtain

\[
(52) \quad x = -\frac{2u}{u^2 + v^2 + 1}, \quad y = -\frac{2v}{u^2 + v^2 + 1}, \quad z = -\frac{2}{u^2 + v^2 + 1}.
\]

Eliminating the parameters \( u \) and \( v \), we obtain

\[
(53) \quad x^2 + y^2 + (z + 1)^2 = 1.
\]

At the same time equations (48) and (49) reduce to

\[
x = uz, \quad y = vz.
\]

Therefore, the unit sphere, tangent to the \( xy \)-plane at the origin, is a surface on which there exists a conjugate system with indeterminate axis curves. The pencil of planes through the \( x \)-axis cuts the sphere in the curves \( v = \text{const} \), and the pencil of planes through the \( y \)-axis cuts the sphere in the curves \( u = \text{const} \). The cones \( S_z \) and \( S_u \) are simply these two coordinate axes. And the axis of every point on the sphere passes through the origin.
The generalization to quadric surfaces is immediate. *At any point* $P$ on any quadric surface select any two tangent lines separating harmonically the two generators through $P$. With each of these tangents as axis, construct a pencil of planes. The planes of these two pencils cut the quadric in two one-parameter families of plane curves which constitute a conjugate system with indeterminate axis curves. The axis of every point on the quadric, defined with reference to this system, passes through $P$.

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