GRATINGS IN THEORY AND PRACTICE. *

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PART I.

It is not my object to treat the theory of diffraction in general but only to apply the simplest ordinary theory to gratings made by ruling grooves with a diamond on glass or metal. This study I at first made with a view of guiding me in the construction of the dividing engine for the manufacture of gratings, and I have given the present theory for years in my lectures. As the subject is not generally understood in all its bearings I have written it for publication.

Let \( p \) be the virtual distance reduced to vacuo through which a ray moves. Then the effect at any point will be found by the summation of the quantity

\[
A \cos b(p - Vt) + B \sin b(p - Vt)
\]

in which \( b = \frac{2\pi}{l} \), \( l \) being the wave-length, \( V \) is the velocity reduced to vacuo, and \( t \) is the time. Making \( \theta = \tan^{-1} \frac{A}{B} \) we can write this

\[
\sqrt{A^2 + B^2} \sin [\theta + b(p - Vt)].
\]

The energy or intensity is proportional to \((A^2 + B^2)\)

Taking the expression

\[
(A + iB)e^{-ib(p - Vt)},
\]

when \( i = \sqrt{-1} \), its real part will be the previous expression for the displacement. Should we use the exponential expression instead of the circular function in our summation we see that we can always obtain the intensity of the light by multiplying the final result by itself with \(-i\) in place of \(+i\), because we have

\[
(A + iB)e^{-ib(p - Vt)} \times (A - iB)e^{ib(p - Vt)} = A^2 + B^2
\]

* Communicated by the author.
† I am much indebted to Dr. Ames for looking over the proofs of this paper and correcting some errors. In the paper I have, in order to make it complete, given some results obtained previously by others, especially by Lord Rayleigh. The treatment is, however, new, as well as many of the results. My object was originally to obtain some guide to the effect of errors in gratings so that in constructing my dividing engine I might prevent their appearance if possible.
Gratings in Theory and Practice.

In cases where a ray of light falls on a surface where it is broken up, it is not necessary to take account of the change of phase at the surface but only to sum up the displacement as given above.

In all our problems let the grating be rather small compared with the distance of the screen receiving the light so that the displacements need not be divided into their components before summation.

Let the point \( x', y', z' \) be the source of light, and at the point \( x, y, z \) let it be broken up and at the same time pass from a medium of index of refraction \( I' \) to one of \( I \). Consider the disturbance at a point \( x'', y'', z'' \) in the new medium. It will be

\[ e^{-ib[I'' + \rho'' - \nu]} \]

where

\[ \rho^2 = x''^2 + y''^2 + z''^2 + x^2 + y^2 + z^2 - 2(xx'' + yy'' + zz''). \]

\[ \rho^2 = x'^2 + y'^2 + z'^2 + x^2 + y^2 + z^2 - 2(xx' + yy' + zz'). \]

Let the point \( x, y, z \) be near the origin of co-ordinates as compared with \( x', y', z' \) or \( x'', y'', z'' \) and let \( \alpha, \beta, \gamma \) and \( \alpha', \beta', \gamma' \) be the direction cosines of \( \rho \) and \( \rho \). Then, writing

\[ R = \sqrt{x'^2 + y'^2 + z'^2} + \sqrt{x''^2 + y''^2 + z''^2} \]

\[ \lambda = I\alpha + I'\alpha' \]

\[ \mu = I\beta + I'\beta' \]

\[ \nu = I\gamma + I'\gamma' \]

we have, for the elementary displacement,

\[ e^{-ib[I - \nu - \lambda x - \mu y - \nu z + i\kappa r]} \]

where \( \kappa = \frac{1}{2} \left[ \frac{I'}{\sqrt{x'^2 + y'^2 + z'^2}} + \frac{I}{\sqrt{x''^2 + y''^2 + z''^2}} \right] \)

and \( r^2 = x^2 + y^2 + z^2 \).

This equation applies to light in any direction. In the special case of parallel light, for which \( \kappa = 0 \), falling on a plane grating with lines in the direction of \( z \), one condition will be that this expression must be the same for all values of \( z \).

Hence \( \nu = 0 \).

If \( N \) is the order of the spectrum and \( a \) the grating space we shall see further on that we also have the condition

\[ ba\mu = 2\pi N = \frac{2\pi a}{I} \mu \]

The direction of the diffracted light will then be defined by the equations
\[\alpha^2 + \beta^2 + \gamma^2 = 0\]
\[I\gamma + I'\gamma' = 0\]
\[I\beta + I'\beta' = \frac{l}{a} N\]

Whence
\[I'a' = I\sqrt{\alpha^2 + 2\frac{l}{a} N\beta - \frac{N^2}{a^2}}\]
\[I'\beta' = \frac{l}{a} N - I\beta\]
\[I'\gamma' = -I\gamma\]

In the ordinary case where the incident and diffracted rays are perpendicular to the lines of the grating, we can simplify the equations somewhat.

Let \(\varphi\) be the angle of incidence and \(\psi\) of diffraction as measured from the positive direction of \(X\).

\[I = I'\cos \varphi + I \cos \psi\]
\[\frac{l}{a} N = \mu = I' \sin \varphi + I \sin \psi\]
\[b = \frac{2\pi}{l}\]

where \(l\) is the wave-length in vacuo.

In case of the reflecting grating \(I = I'\) and we can write

\[\lambda = I' \cos \varphi + I \cos \psi\]
\[\frac{l}{a} N = \mu = I' \sin \varphi + I \sin \psi\]

This is only a very elementary expression as the real value would depend on the nature of the obstacle, the angles, etc., but it will be sufficient for our purpose.

The disturbance due to any grating or similar body will then be very nearly

\[\int \int e^{-ib[R - Vt - \lambda x - \mu y - \nu z + \kappa(x^2 + y^2 + z^2)]} \, ds.\]

where \(ds\) is a differential of the surface. For parallel rays, \(\kappa = 0\).

**Plane Gratings.**

In this case the integration can often be neglected in the direction of \(z\) and we can write for the disturbance in case of parallel rays,

\[e^{-ib(R - Vt)} \int \int e^{-ib[-\lambda x - \mu y]} \, ds.\]
CASE I.—SIMPLE PERIODIC RULING.

Let the surface be divided up into equal parts in each of which one or more lines or grooves are ruled parallel to the axis of z.

The integration over the surface will then resolve itself into an integration over one space and a summation with respect to the number of spaces. For in this case we can replace y by \( na + y \) where a is the width of a space and the displacement becomes

\[
e^{-ib(R - Vt)} \sum e^{ib\mu an} \int e^{ib(\lambda x + \mu y)} ds
\]

but

\[
\sum_{n=1}^{N} e^{ib\mu an} = e^{ib \frac{n-1}{2} \mu a} \frac{\sin \frac{b a \mu}{2}}{\sin \frac{b a \mu}{2}}
\]

Multiplying the disturbance by itself with \(-i\) in place of \(+i\) we have for the light intensity

\[
\left[ \sin \frac{b a \mu}{2} \right]^2 \left[ \int e^{-ib(\lambda x + \mu y)} ds \right] \left[ \int e^{ib(\lambda x + \mu y)} ds \right]
\]

The first term indicates spectral lines in positions given by the equation

\[
\sin \frac{b a \mu}{2} = 0
\]

with intensities given by the last integral. The intensity of the spectral lines then depends on the form of the groove as given by the equation \( x = f(y) \) and upon the angles of incidence and diffraction. The first factor has been often discussed and it is only necessary to call attention to a few of its properties.

When \( b a \mu = 2\pi N \), N being any whole number, the expression becomes \( n^2 \). On either side of this value the intensity decreases until \( n b a \mu' = 2\pi N \), when it becomes 0.

The spectral line then has a width represented by \( \mu' - \mu'' = \frac{2\mu}{n} \) nearly; on either side of this line smaller maxima exist too faintly to be observed. When two spectral lines are nearer together than half their width, they blend and form one line. The defining power of the spectroscope can be expressed in terms of the quotient of the wave-length by the difference of wave-length of two lines that can just be seen as divided. The defining power is, then,
Now \( na \) is the width of the grating. Hence, using a grating at a given angle, the defining power is independent of the number of lines to the inch and only depends on the width of the grating and the wave-length. According to this, the only object of ruling many lines to the inch in a grating is to separate the spectra so that, with a given angle, the order of spectrum shall be less.

Practically the gratings with few lines to the inch are much better than those with many, and hence have better definition at a given angle than the latter except that the spectra are more mixed up and more difficult to see.

It is also to be observed that the defining power increases with shorter wave-lengths, so that it is three times as great in the ultra-violet as in the red of the spectrum. This is of course the same with all optical instruments such as telescopes and microscopes.

The second term which determines the strength of the spectral lines will, however, give us much that is new.

First let us study the effect of the shape of the groove on the brightness. If \( N \) is the order of the spectrum and \( a \) the grating space we have

\[
\mu = I(\sin \phi + \sin \psi) = \frac{Nl}{a}
\]

since \( \sin \frac{ba \mu}{2} = 0 \)

and the intensity of the light becomes proportional to

\[
\left[ \int \int e^{i2\pi\left(\frac{\lambda}{a}x + \frac{N}{a}\right)} \, ds \right] \left[ \int \int e^{-i2\pi\left(\frac{\lambda}{a}x + \frac{N}{a}\right)} \, ds \right]
\]

It is to be noted that this expression is not only a function of \( N \) but also of \( l \), the wave-length. This shows that the intensity in general may vary throughout the spectrum according to the wave-length and that the sum of the light in any one spectrum is not always white light.

This is a peculiarity often noticed in gratings. Thus one spectrum may be almost wanting in the green, while another may contain an excess of this color; again there may be very little blue in one spectrum while very often the similar spectrum on the other side may have its own share and that of the other one also. For this reason I have found it almost impossible to predict what the ultra red spectrum may be, for it is often weak even where the visible spectrum is strong.

\* An expression of Lord Rayleigh's.
The integral may have almost any form although it will naturally tend to be such as to make the lower orders the brightest when the diamond rules a single and simple groove. When it rules several lines or a compound groove, the higher orders may exceed the lower in brightness and it is mathematically possible to have the grooves of such a shape that, for given angles, all the light may be thrown into one spectrum.

It is not uncommon, indeed very easy, to rule gratings with immensely bright first spectra, and I have one grating where it seems as if half the light were in the first spectrum on one side. In this case there is no reflection of any account from the grating held perpendicularly: indeed to see one's face, the plate must be held at an angle, in which case the various features of the face are seen reflected almost as brightly as in a mirror but drawn out into spectra. In this case all the other spectra and the central image itself are very weak.

In general it would be easy to prove from the equation that want of symmetry in the grooves produces want of symmetry in the spectra, a fact universally observed in all gratings and one which I generally utilize so that the light may be concentrated in a few spectra only.

**Example I. Square Grooves.**

When the light falls nearly perpendicularly on the plate, we need not take the sides into account but only sum up the surface of the plate and the bottom of the groove. Let the depth be \( \frac{a}{m} \) and the width equal to \( \frac{a}{m} \).

The intensity then becomes proportional to

\[
\frac{1}{N^2} \sin^2 \pi \frac{N}{m} \sin^2 \pi \frac{\lambda}{l} X
\]

This vanishes when

\[
N = m, 2m, 3m, \text{etc.}
\]

or,

\[
\frac{\lambda X}{l} = 0, 1, 2, 3, \text{etc.}
\]

The intensity of the central light, for which \( N = 0 \), will be

\[
\frac{\pi^2}{m^2} \sin^2 \left( \pi \frac{\lambda}{l} X \right).
\]

This can be made to vanish for only one angle for a given wave-length. Therefore, the central image will be colored and
the color will change with the angle, an effect often observed in actual gratings. The color ought to change, also, on placing the grating in a liquid of different index of refraction since \( \lambda \) contains \( I \), the index of refraction.

It will be instructive to take a special case, such as light falling perpendicularly on the plate. For this case

\[ \varphi = 0, \; \lambda = I(1 + \cos \psi) \] and \( \mu = I \sin \psi = \frac{NI}{a} \).

Hence, \( \lambda = I \left\{ 1 + \sqrt{1 - \left( \frac{NI}{a} \right)^2} \right\} \).

The last term in the intensity will then be

\[ \sin^2 \left\{ \pi I \left[ \frac{1}{I} + \sqrt{\frac{1}{I^2} - \left( \frac{N}{a} \right)^2} \right] \right\} \]

As an example, let the green of the second order vanish. In this case, \( I = .00005 \). \( N = 2 \). Let \( a = .0002 \) c.m. and \( I = 1 \).

Then, \( X \left[ 20000 + \sqrt{(20000)^2 - (10000)^2} \right] = n. \)

Whence, \( X = \frac{n}{37300} \).

where \( n \) is any whole number. Make it 1.

Then the intensity, as far as this term is concerned, will be as follows:

<table>
<thead>
<tr>
<th>Minima where Intensity is 0</th>
<th>Maxima where Intensity is 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wave-lengths</td>
<td>Wave-lengths</td>
</tr>
<tr>
<td>1st spec.</td>
<td>1st spec.</td>
</tr>
<tr>
<td>.00000526</td>
<td>.0001000</td>
</tr>
<tr>
<td>.00000268</td>
<td>.000003544</td>
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<tr>
<td>.0001600</td>
<td>.00002137</td>
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<tr>
<td>2nd</td>
<td>2nd</td>
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<tr>
<td>.00000500</td>
<td>.00000833</td>
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<td>.0000266</td>
<td>.000003463</td>
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<tr>
<td>.00000853</td>
<td>.00002119</td>
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<tr>
<td>3rd</td>
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<td>.00000462</td>
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<td>.0000283</td>
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<td>.00000651</td>
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<td>4th</td>
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<td>.00000499</td>
<td>.000002050</td>
</tr>
<tr>
<td>5th</td>
<td>etc.</td>
</tr>
<tr>
<td>etc.</td>
<td>etc.</td>
</tr>
</tbody>
</table>

The central light will contain the following wave-lengths as a maximum:

\[ .0001072 \quad .00003575 \quad .0000214, \text{ etc.} \]

Of course it would be impossible to find a diamond to rule a rectangular groove as above and the calculations can only be looked upon as a specimen of innumerable light distributions according to the shape of groove.

Every change in position of the diamond gives a different light distribution and hundreds of changes may be made every day and yet the same distribution will never return, although one may try for years.
Example 2.—Triangular Groove.

Let the space $a$ be cut into a triangular groove, the equations of the sides being $x = -cy$, and $x = c'(y - a)$, the two cuttings coming together at the point $y = u$. Hence we have $-cu = c'(u - a)$, and $ds = dy\sqrt{1 + c'^2}$, or $dy\sqrt{1 + c^2}$. Hence the intensity is proportional to

$$I^2 \left\{ \frac{1 + c^2}{(\mu - c\lambda)^2} \sin^2 \pi u(\mu - c\lambda) + \frac{1 + c'^2}{(\mu + c'\lambda)^2} \sin^2 \pi (u - a)(\mu + c'\lambda) \right. \right.$$

$$+ \frac{\sqrt{(1 + c^2)(1 + c'^2)}}{(\mu - cy)(\mu + c'\lambda)} \sin \frac{\pi u(\mu - c\lambda)}{I} \sin \frac{\pi (u - a)(\mu + c'\lambda)}{I} \left. \right\} \cos \frac{\pi}{I} [(\mu + c'\lambda)(a - u) - n(\mu - c\lambda)]$$

This expression is not symmetrical with respect to the normal to the grating, unless the groove is symmetrical, in which case $c = c'$ and $u = \frac{a}{2}$.

In this case, as in the other, the colors of the spectrum are of variable intensity, and some of them may vanish as in the first example, but the distribution of intensity is in other respects quite different.

Case 2.—Multiple Periodic Ruling.

Instead of having only one groove ruled on the plate in this space $a$, let us now suppose that a series of similar lines are ruled. We have, then, to obtain the displacement by the same expression as before, that is

$$\sin \frac{na}{2} \int e^{ib(\lambda u + cy)} ds,$$

except that the last integral will extend over the whole number of lines ruled within the space $a$.

In the spaces $a$ let a number of equal grooves be ruled commencing at the points $y = 0, y_1, y_2, y_3, \text{etc.}$, and extending to the points $w, y_1 + w, y_2 + w, \text{etc.}$ The surface integral will then be divided into portions from $w$ to $y_1$, from $y_1 + w$ to $y_2$, etc., on the original surface of the plate for which $x = 0$, and from $w$ to $0$, from $y_1 + w$ to $y_1$, etc., for the grooves.

The first series of integrals will be
\[
\int e^{ib \mu y} dy = \left\{ \frac{1}{ib \mu} \right\} e^{ib \mu w} + \left( 1 - e^{ib \mu w} \right) \left( e^{ib \mu y_1} + e^{ib \mu y_2} + \text{etc.} \right)
\]
\[
= \frac{1}{ib \mu} \left( -e^{ib \mu w} + \left( 1 - e^{ib \mu w} \right) \left( e^{ib \mu y_1} + e^{ib \mu y_2} + \text{etc.} \right) + e^{ib \mu a} \right)
\]

But, \( e^{ib \mu a} = 1 \) since \( b \mu a = 0 \) for any maximum, and thus the integral becomes
\[
\frac{1 - e^{ib \mu w}}{ib \mu} \left( 1 + e^{ib \mu y_1} + e^{ib \mu y_2} + \text{etc.} \right)
\]

The second series of integrals will be
\[
\int_0^w e^{ib(\lambda n + \mu y)} ds \left( 1 + e^{ib \mu y_1} + \text{etc.} \right)
\]

The total integral will then be
\[
\sin \frac{n}{2} \sin \frac{ba \mu}{n} \left[ \frac{1 - e^{ib \mu w}}{ib \mu} + \int_0^w e^{ib(\lambda x + \mu y)} ds \right] \left[ 1 + e^{ib \mu y_1} + e^{ib \mu y_2} + \text{etc.} \right]
\]

As before, multiply this by the same with the sign of \( i \) changed to get the intensity.

**Example I.—Equal Distances.**

The space, \( a \), contains \( n' - 1 \) equidistant grooves, so that
\[
y_1 = y_2 - y_1 = \text{etc.,} = \frac{a}{n'}
\]

Hence the displacement becomes
\[
\sin \frac{n}{2} \sin \frac{ba \mu}{2n'} \left[ \frac{1 - e^{ib \mu w}}{ib \mu} + \int_0^w e^{ib(\lambda x + \mu y)} ds \right]
\]

As the last term is simply the integral over the space \( \frac{a}{n'} \) in a different form from before, this is a return to the form we previously had except that it is for a grating of \( nn' \) lines instead of \( n \) lines, the grating space being \( \frac{a}{n'} \).
EXAMPLE II.—Two Grooves.

\[ 1 + e^{ib\mu y_1} = 2e^{\frac{ib\mu y_1}{2}} \cos \frac{b\mu y_1}{2} \]

But \( b\mu = 2N\pi \). Hence this becomes

\[ 2e^{i\pi N\frac{y_1}{a}} \cos \pi \frac{N}{a}y_1 \]

The square of the last term is a factor in the intensity. Hence the spectrum will vanish when we have

\[ \frac{N}{a}y_1 = 1, \frac{3}{2}, \frac{5}{2}, \text{etc.} \]

or, \( N = 1, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, y_1, \frac{9}{2}, y_1, \frac{11}{2}, y_1, \text{etc.} \)

Thus when \( \frac{a}{y_1} = 2 \), the 1st, 3d, etc., spectra will disappear making a grating of twice the number of lines to the c.m.

When \( \frac{a}{y_1} = 4 \), the 2d, 6th, 10th, etc, spectra disappear. When \( \frac{a}{y_1} = 6 \), the 3d, 9th, etc., spectra disappear.

The case in which \( \frac{a}{y_1} = 4 \), as Lord Rayleigh has shown, would be very useful as the second spectrum disappears leaving the red of the first and the ultra violet of the third without contamination by the second. In this case two lines are ruled and two left out. This would be easy to do but the advantages would hardly pay for the trouble owing to the following reasons: Suppose the machine was ruling 20,000 lines to the inch. Leaving out two lines and ruling two would reduce the dispersion down to a grating with 5,000 lines to the inch. Again, the above theory assumes that the grooves do not overlap. Now I believe that in nearly, if not all, gratings with 20,000 lines to the inch the whole surface is cut away and the grooves overlap. This would cause the second spectrum to appear again after all our trouble.

Let the grooves be nearly equidistant, one being slightly displaced. In this case \( y_1 = \frac{a}{2} + v \).

\[ \cos^2 \pi \frac{N}{a}y_1 = \left( \cos \frac{\pi N}{2} \cos \frac{\pi N}{a}v - \sin \frac{\pi N}{2} \sin \frac{\pi N}{a}v \right)^2 \]

For the even spectra this is very nearly unity, but for the odd it becomes

* A theorem of Lord Rayleigh's.
Hence the grating has its principal spectra like a grating of space \( \frac{a}{2} \), but there are still the intermediate spectra due to the space \( a \), and of intensities depending on the squares of the order of spectrum, and the squares of the relative displacement, a law which I shall show applies to the effect of all errors of the ruling.

This particular effect was brought to my attention by trying to use a tangent screw on the head of my dividing engine to rule a grating with say 28,872 lines to the inch, when a single tooth gave only 14,436 to the inch. However carefully I ground the tangent screw I never was able to entirely eliminate the intermediate spectra due to 14,436 lines, and make a pure spectrum due to 28,872 lines to the inch, although I could nearly succeed.

**Example 3.—One Groove in \( m \) Misplaced.**

Let the space \( a \) contain \( m \) grooves equidistant except one which is displaced a distance \( v \). The displacement is now proportional to

\[
1 + e^{ib\mu \frac{a}{2m}} + e^{2ib\mu \frac{a}{2m}} + \text{etc.} + e^{ib\mu (p\frac{a}{m} + v)} + \text{etc.} + e^{ib\mu \frac{m-1}{m} a}
\]

\[
= e^{ib\mu \frac{m-1}{2m} a} \left\{ \frac{\sin \frac{b\mu a}{2}}{\sin \frac{b\mu a}{2m}} + ib\mu v e^{ib\mu 2p - m + 1} \frac{2p - m + 1}{2m} \right\}
\]

Multiplying this by itself with \(-i\) in place of \(+i\), and adding the factors in the intensity, we have the whole expression for the intensity. One of the terms entering the expression will be

\[
\frac{\sin n \frac{b\mu a}{2}}{\sin \frac{b\mu a}{2m}} \sin \frac{b\mu a}{2} \cdot \sin \frac{b\mu a}{2} \frac{2p - m + 1}{m}
\]

Now the first two terms have finite values only around the points \( \frac{ba\mu}{2} = mN\pi \), where \( mN \) is a whole number. But \( 2p - m + 1 \) is also a whole number, and hence the last term is zero at these points. Hence the term vanishes and leaves the intensity, omitting the groove factor,
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\[
\sin^2 n \frac{b\alpha x}{2} + \sin^2 n \frac{b\alpha x}{2m} \frac{(b\mu y)^2}{(b\mu y)^2} + \sin^2 n \frac{b\alpha x}{2} \frac{(b\mu y)^2}{(b\mu y)^2}
\]

The first term gives the principle spectra as due to a grating space of \( \frac{a}{m} \) and number of lines \( nm \) as if the grating were perfect. The last term gives entirely new spectra due to the grating space, \( a \), and with lines of breadth due to a grating of \( n \) lines and intensities equal to \( (b\mu y)^2 \).

Hence, when the tangent screw is used on my machine for 14,436 lines to the inch, there will still be present weak spectra due to the 14,436 spacing although I should rule say 400 lines to the \( mm \). This I have practically observed also.

The same law holds as before that the relative intensity in these subsidiary spectra varies as the square of the order of the spectrum and the square of the deviation of the line, or lines from their true position.

So sensitive is a dividing engine to periodic disturbances that all the belts driving the machine must never revolve in periods containing an aliquot number of lines of the grating; otherwise they are sure to make spectra due to their period.

As a particular case of this section we have also to consider

**Periodic Errors of Ruling.—Theory of “Ghosts.”**

In all dividing engines the errors are apt to be periodic due to “drunken” screws, eccentric heads, imperfect bearings, or other causes. We can then write

\[ y = n_1 a_1 + a_2 \sin(e_1 n) + a_3 \sin(e_2 n) + \text{etc.} \]

The quantities \( e_1, e_2, \text{etc.,} \) give the periods, and \( a_1, a_2, \text{etc.,} \), the amplitudes of the errors. We can then divide the integral into two parts as before, an integral over the groove and spaces and a summation with respect to the numbers.

\[
\sum \int_{y'}^{y''} e^{-ib(\lambda x + \mu y)} ds = \sum \int_{y'}^{y''} e^{-ib\mu y} \int_0^{y''} e^{-ib(\lambda n + \mu y)} ds
\]

It is possible to perform these operations exactly, but it is less complicated to make an approximation, and take \( y'' - y' = a \), a constant as it is very nearly in all gratings. Indeed the error introduced is vanishingly small. The integral which depends on the shape of the groove, will then go outside the summation sign and we have to perform the summation.
Let $J_n$ be a Bessel's function. Then

$$\cos(u \sin \varphi) = J_0(u) + 2[J_1(u) \cos \varphi + J_3(u) \cos^3 \varphi + \text{etc.}]$$

$$\sin (u \sin \varphi) = 2(J_1(u) \sin \varphi + J_3(u) \sin^3 \varphi + \text{etc.})$$

But $e^{-iu \sin \varphi} = \cos(u \sin \varphi) - i \sin(u \sin \varphi).$

Hence the summation becomes

$$\sum e^{-i b \mu a_n} \left[ J_0(b \mu a_n) + 2(-i J_1(b \mu a_n) \sin e_n + J_3(b \mu a_n) \cos 2e_n - \text{etc.}) \right]$$

$$\times \left[ J_0(b \mu a_n) + 2(-i J_1(b \mu a_n) \sin e_n + J_3(b \mu a_n) \cos 2e_n - \text{etc.}) \right]$$

$$\times \left[ J_0(b \mu a_n) + \text{etc.} \right]$$

**CASE I.—SINGLE PERIODIC ERROR.**

In this case only $a_n$ and $a_1$ exist. We have the formula

$$\sum_{n=0}^{n-1} e^{-ipn} = e^{-i \frac{\pi n}{2}} \frac{\sin \frac{p n}{2}}{\sin \frac{p}{2}}$$

Hence the expression for the intensity becomes

$$\left\{ \frac{\sin n \frac{b \mu a_n}{2}}{\sin \frac{b \mu a_n}{2}} \right\}^2 + \left\{ \frac{\sin \frac{b \mu a_n + e_1}{2}}{\sin \frac{b \mu a_n + e_1}{2}} \right\}^2$$

$$+ \left\{ \frac{\sin \frac{b \mu a_n - e_1}{2}}{\sin \frac{b \mu a_n - e_1}{2}} \right\}^2 + \text{etc.}$$

As $n$ is large, this represents various very narrow spectral lines whose light does not overlap and thus the different terms are independent of each other. Indeed in obtaining this expression the products of quantities have been neglected for this reason because one or the other is zero at all points. These lines are all alike in relative distribution of light and their intensities and positions are given by the following table.
Gratings in Theory and Practice.

<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>$\mu = \frac{2\pi N}{ba_v}$</td>
<td>$J_{\mu}^2(b\mu a_i)$</td>
<td>Primary lines.</td>
</tr>
<tr>
<td>$\mu_1 = \mu \pm \frac{e_1}{ba_v}$</td>
<td>$J_{\mu}^2(b\mu a_i)$</td>
<td>Ghosts of 1st order.</td>
</tr>
<tr>
<td>$\mu_2 = \mu \pm \frac{2e_1}{ba_v}$</td>
<td>$J_{\mu}^2(b\mu a_i)$</td>
<td>Ghosts of 2nd order.</td>
</tr>
<tr>
<td>$\mu_3 = \mu \pm \frac{3e_1}{ba_v}$</td>
<td>$J_{\mu}^2(b\mu a_i)$</td>
<td>Ghosts of 3rd order.</td>
</tr>
<tr>
<td>etc.</td>
<td>etc.</td>
<td>etc.</td>
</tr>
</tbody>
</table>

Hence the light which would have gone into the primary line now goes to making the ghosts, so that the total light in the line and its ghosts is the same as in the original without ghosts.

The relative intensities of the ghosts as compared with the primary line is

$$\frac{J_{\mu}^2(b\mu a_i)}{J_{\mu}^2(b\mu a_1)}$$

This for very weak ghosts of the first, second, third, etc., order, becomes

$$\left(\frac{\pi N a_i}{a_o}\right)^2, \frac{1}{2}\left(\frac{\pi N a_i}{a_o}\right)^4, \frac{1}{6}\left(\frac{\pi N a_i}{a_o}\right)^6, \text{etc.}$$

The intensity of the ghosts of the first order varies as the square of the order of the spectrum and as the square of the relative displacement as compared with the grating space $a_o$. This is the same law as we before found for other errors of ruling, and it is easy to prove that it is general. Hence

The effect of small errors of ruling is to produce diffused light around the spectral lines. This diffused light is subtracted from the light of the primary line, and its comparative amount varies as the square of the relative error of ruling and the square of the order of the spectrum.

Thus the effect of the periodic error is to diminish the intensity of the ordinary spectral lines (primary lines) from the intensity 1 to $J_{\mu}^2(b\mu a_i)$, and surround it with a symmetrical system of lines called ghosts, whose intensities are given above.

When the ghosts are very near the primary line, as they nearly always are in ordinary gratings ruled on a dividing engine with a large number of teeth in the head of the screw, we shall have

$$J_{\mu}^2 ba_i\left(\mu \pm \frac{e_1}{ba_v}\right) + J_{\mu}^2 ba_i\left(\mu - \frac{e_1}{ba_v}\right) = 2J_{\mu}^2 ba_i \mu \text{ nearly.}$$
Hence the total light is by a known theorem,

\[ J_0^2 + 2[J_1^2 + J_2^2 + \text{etc.}] = 1. \]

Thus, in all gratings, the intensity of the ghosts as well as the diffused light increases rapidly with the order of the spectrum. This is often marked in gratings showing too much crystalline structure. For the ruling brings out the structure and causes local difference of ruling which is equivalent to error of ruling as far as diffused light is concerned.

For these reasons it is best to get defining power by using broad gratings and a low order of spectra although the increased perfection of the smaller gratings makes up for this effect in some respects.

There is seldom advantage in making both the angle of incidence and diffraction more than 45°, but, if the angle of incidence is 0, the other angle may be 60°, or even 70°, as in concave gratings. Both theory and practice agree in these statements.

Ghosts are particularly objectionable in photographic plates, especially when they are exposed very long. In this case ghosts may be brought out which would be scarcely visible to the eye.

As a special case, take the following numerical results:

<table>
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<tr>
<th>( N )</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tbody>
<tr>
<td>( \frac{a_1}{a_o} )</td>
<td>25'</td>
<td>50'</td>
<td>100</td>
</tr>
<tr>
<td>( \frac{\pi N a_1}{a_o} )</td>
<td>63'</td>
<td>252'</td>
<td>1008</td>
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</table>

In a grating with 20,000 lines to the inch, using the third spectrum, we may suppose that the ghosts corresponding to \( \frac{a_1}{a_o} = \frac{1}{50} \) will be visible and those for \( \frac{a_1}{a_o} = \frac{1}{25} \) very troublesome. The first error is \( a_1 = \frac{100,000}{9} \) in. and the second \( a_1 = \frac{50,000}{9} \) in. Hence a periodic displacement of one millionth of an inch will produce visible ghosts and one five hundred thousandth of an inch will produce ghosts which are seen in the second spectrum and are troublesome in the third. With very bright spectra these might even be seen in the first spectrum. Indeed an over exposed photographic plate would readily bring them out.

When the error is very great, the primary line may be very faint or disappear altogether, the ghosts to the number of
Gratings in Theory and Practice.

twenty or fifty or more being often more prominent than the original line. Thus, when

\[ b\mu a_1 = 2.405, 5.52, 8.65 \text{ etc.} = 2\pi a_1 \]

the primary line disappears. When

\[ b\mu a_1 = 0, 3.83, 7.02 \text{ etc.} = 2\pi a_1 \]

the ghosts of the first order will disappear. Indeed we can make any ghost disappear by the proper amount of error.

Of course, in general

\[ J_n = \frac{2(n-1)}{v} J_{n-1} - J_{n-2}. \]

Thus a table of ghosts can be formed readily and we may always tell when the calculation is complete by taking the sum of the light and finding unity.

| \( \frac{2\pi a_1}{a_0} \) | \( J_0 \) | \( J_1 \) | \( J_2 \) | \( J_3 \) | \( J_4 \) | \( J_5 \) | \( J_6 \) | \( J_7 \) | \( J_8 \) | \( J_9 \) | \( J_{10} \) | \( J_{11} \) | \( J_{12} \) | \( J_{13} \) | \( J_{14} \)
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<td>0.022</td>
<td>0.011</td>
<td>0.009</td>
<td>0.022</td>
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</table>

This table shows how the primary line weakens as the periodic error increases, becoming 0 at

\[ 2\pi a_0 = 2.405. \]

It then strengthens and weakens periodically, the greatest strength being transferred to one of the ghosts of higher and higher order as the error increases.

Thus one may obtain an estimate of the error from the appearance of the ghost.

Some of these wonderful effects with 20 to 50 ghosts stronger than the primary line I have actually observed in a grating ruled on one of my machines before the bearing end of the screw had been smoothed. The effect was very similar to these calculated results.


Henry A. Rowland.

Double Periodic Error.

Supposing as before that there is no overlapping of the lines, we have the following:

\[ \mu = \frac{2\pi N}{ba} \]

**Places.**

\[ \mu_1 = \mu \pm \frac{e_1}{ba} \]

\[ \mu_2 = \mu \pm \frac{e_2}{ba} \]

\[ \mu_3 = \mu \pm \frac{e_1 \pm e_2}{ba} \]

\[ \mu_4 = \mu \pm \frac{2e_1}{ba} \]

\[ \mu_5 = \mu \pm \frac{2e_2}{ba} \]

\[ \mu_6 = \mu \pm \frac{e_1 \pm 2e_2}{ba} \]

**Intensities.**

\[ \left[ J_0(ba_1) J_1(ba_2) \right]^2 \]

Primary line.

\[ \left[ J_0(ba_1) J_1(ba_2) \right]^2 \]

Ghosts of 1st order.

\[ \left[ J_0(ba_1) J_1(ba_2) \right]^2 \]

Ghosts of 2nd order.

\[ \left[ J_0(ba_1) J_1(ba_2) \right]^2 \]

Ghosts of 3rd order.

etc.

Each term in this table of ghosts simply expresses the fact that each periodic error produces the same ghosts in the same place as if it were the only error, while others are added which are the ghosts of ghosts. The intensities, however, are modified in the presence of these others.

Writing \( c_1 = ba_1 \) and \( c_2 = ba_2 \),

The total light is

\[ J_0^2(c_1) J_0^2(c_2) + \left\{ 2J_0^2(c_1) J_1^2(c_2) \right\} + \left\{ 2J_0^2(c_2) J_1^2(c_1) \right\} + \left\{ 4J_0^2(c_1) J_1^2(c_2) \right\} + \text{etc.} \]

which we can prove to be equal to 1.
Gratings in Theory and Practice.

Hence the sum of all the light is still unity, a general proposition which applies to any number of errors.

The positions of the lines when there is any number of periodic errors can always be found by calculating first the ghosts due to each error separately; then the ghosts due to these primary ghosts for it as if it were the primary line, and so on ad infinitum.

In case the ghosts fall on top of each other the expression for the intensity fails. Thus when \( e_2 = 2e_1, e_3 = 3e_1, \) etc., the formula will need modification. The positions are in this case only those due to a single periodic error, but the intensities are very different.

We have hitherto considered cases in which the error could not be corrected by any change of focus in the objective. It is to be noted, however, that for any given angle and focus, every error of ruling can be neutralized by a proper error of the surface, and that all the results we have hitherto obtained for errors of ruling can be produced by errors of surface, and many of them by errors in size of groove cut by the diamond. Thus ghosts are produced not only by periodic errors of ruling but by periodic waves in the surface, or even by a periodic variation in the depth of ruling. In general, however, a given solution will apply only to one angle and, consequently, the several results will not be identical; in some cases, however, they are perfectly so.

Let us now take up some cases in which change of focus can occur. The term \( n r^2 \) in the original formula must now be retained.

Let the lines of the grating be parallel to each other. We can then neglect the terms in \( z \) and can write \( r^2 = y^2 \) very nearly. Hence the general expression becomes

\[
\int e^{ib(-\lambda x + \kappa y - ny^2)} ds,
\]

where \( \kappa \) depends on the focal length. This is supposed to be very large, and hence \( \kappa \) is small.

This integral can be divided into two parts, an integral over the groove and the intervening space, and a summation for all the grooves. The first integral will slightly vary with change in the distance of the grooves apart, but this effect is vanishingly small compared with the effect on the summation, and can thus be neglected. The displacement is thus proportional to

\[
\Sigma e^{ib(\kappa y - ny^2)}
\]
CASE 1.—LINES AT VARIABLE DISTANCES.

In this case we can write in general

\[ y = an + a_n n^2 + a_n n^3 + \text{etc.} \]

As \( x, a, a_n, \text{etc.} \), are small, we have for the displacement, neglecting the products of small quantities,

\[ \sum e^{ib\mu(a_n n^2 + a_n n^3 + \text{etc.})} \]

Hence the term \( a_n n^3 \) can be neutralized by a change of forms expressed by \( \mu a = \kappa a^2 \). Thus a grating having such an error will have a different focus according to the angle \( n \), and the change will be + on one side and — on the other.

This error often appears in gratings and, in fact, few are without it.

A similar error is produced by the plate being concave, but it can be distinguished from the above error by its having the focus at the same angle on the two sides the same instead of different.

According to this error, \( a_n n^3 \), the spaces between the lines from one side to the other of the grating, increase uniformly in the same manner as the lines in the B group of the solar spectrum are distributed. Fortunately it is the easiest error to make in ruling, and produces the least damage.

The expression to be summed can be put in the form

\[ \sum e^{ib\mu n}\left[1 + ib(\mu a_1 - \kappa a^2)n + ib\mu a_1 n + ib[\mu a + ib(\mu a_1 - \kappa a^2)]n + \text{etc.}\right] \]

The summation of the different terms can be obtained as shown below, but, in general, the best result is usually sought by changing the focus. This amounts to the same as varying \( \kappa \) until \( \mu a_1 - \kappa a^2 = 0 \) as before. For the summation we can obtain the following formula from the one already given. Thus

\[ \sum_{n=0}^{n-1} e^{2ipn} = \frac{\sin np}{np} e^{-ip(n-1)} \]

Hence

\[ \sum_{n=0}^{n-1} n^m e^{2ipn} = \frac{1}{(2i)^m} e^{ip(n-1)} \left( \frac{d}{dp} + i(n - 1) \right)^m \frac{\sin np}{np} \]

When \( n \) is very large, writing \( \frac{b\mu n}{2} = \rho n = \pi Nn + q \), we have

\[ \sum_{n=0}^{n-1} n^m e^{2ipn} = \frac{n^m + 1}{(2i)^m} e^{iq} \left( \frac{d}{dq} + i \right)^m \frac{\sin q}{q} \]
Gratings in Theory and Practice.

Whence writing

\[ \begin{align*}
\mathbf{c} &= b(\mu a_1 - \kappa a^2) \\
\mathbf{c}' &= b\mu a_2 \\
\mathbf{c}'' &= b[\mu a_1 + ib(\mu a_1 - \kappa a^2)] \\
\mathbf{c}''' &= \text{etc.}
\end{align*} \]

the summation is

\[ \sum \left( c\frac{n^3}{4} + c'\frac{n^4}{8} + c''\frac{n^5}{16} + \right) \]

\[ + \left( 2c\frac{n^3}{4} + 3c'\frac{n^4}{8} + 4c''\frac{n^5}{16} + \right) \frac{\sin q}{q} \]

\[ - i\left( c\frac{n^3}{4} + 3c'\frac{n^4}{8} + 6c''\frac{n^5}{16} + \right) \frac{\sin q}{q^2} \]

\[ - \left( c'\frac{n^4}{8} + 4c''\frac{n^5}{16} + \right) \frac{\sin q}{q^3} \]

\[ + i\left( c''\frac{n^5}{16} + \right) \frac{\sin q}{q^4} \]

\[ + \text{etc.} \]

\[ \frac{d}{dq} \sin q = \frac{q \cos q - \sin q}{q^2} \]

\[ \frac{d^2}{dq^2} \sin q = -2q \cos q + (2 - q^2) \sin q \]

\[ \frac{d^3}{dq^3} \sin q = \frac{q(6 - q^2) \cos q - (6 - 3q^2) \sin q}{q^4} \]

etc.

These equations serve to calculate the distribution of light intensity in a grating with any error of line distribution suitable to this method of expansion and at any focal length. For this purpose the above summation must be multiplied by itself with + i in place of − i.

The result is for the light intensity

\[ \left\{ \frac{n \sin q}{q} \right\} + \left( 2c\frac{n^3}{4} + 2c'\frac{n^4}{8} + \text{etc.} \right) \frac{d}{dq} \frac{\sin q}{q} \]

\[ - \left( c'\frac{n^4}{8} + 4c''\frac{n^5}{16} + \text{etc.} \right) \frac{d^2}{dq^2} \frac{\sin q}{q} \]

\[ + \left\{ c\frac{n^3}{4} + 3c'\frac{n^4}{8} + \text{etc.} \right\} \frac{d^3}{dq^3} \frac{\sin q}{q} \]

\[ - \left( c''\frac{n^5}{16} + \text{etc.} \right) \frac{d^4}{dq^4} \frac{\sin q}{q} + \text{etc.} \]
Henry A. Rowland.

As might have been anticipated, the effect of the additional terms is to broaden out the line and convert it into a rather complicated group of lines; as can sometimes be observed with a bad grating. At any given angle the same effect can be produced by variation of the plate from a perfect plane. Likewise the effect of errors in the ruling may be neutralized for a given angle by errors of the ruled surface, as noted in the earlier portions of the paper.