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A Verification of the Theory of Brownian Movements and a Direct Determination of the Value of *NE* for Gaseous Ionization

BY

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A VERIFICATION OF THE THEORY OF BROWNIAN MOVEMENTS AND A DIRECT DETERMINATION OF THE VALUE OF *NE* FOR GASEOUS IONIZATION.[1]

BY HARVEY FLETCHER.

INTRODUCTION.

IN 1827 a botanist named Brown noticed that small particles which were suspended in a liquid were in a continual state of agitation; but it was not until 1889 that the correct explanation of this phenomenon was given, when Gouy[2] suggested that it was due to molecular bombardments. In 1905 Einstein[3] deduced a formula which makes it possible to calculate the magnitude of the Brownian displacements. Perrin,[4] Henri,[5] Chandesaignes,[6] and others, working with minute particles suspended in a liquid, obtained results which gave a fair agreement between the calculated and observed values of the displacement. Broglie,[7] who worked with particles of tobacco smoke suspended in air at atmospheric pressure, obtained displacements of the same order of magnitude as given by Einstein's formula.

In the first part of this paper a modification of this formula is proposed, and a new method is given for studying Brownian movements. Formulæ are deduced which give the probable distribution of a great many observed values of:

(1) The displacement of a particle due to Brownian movements.

[1] A brief summary of this paper was given in Science, February 17, 1911, and a more complete abstract in Phys. Zeit., 12, 1911, pp. 202–208.

[2] Journ. de Phys., T. 7, 1888, p. 561.

[3] Ann. de Phys., 4″ Series, T. XVII., p. 549, 1905.

[4] Le Radium, T. 6, p. 353, December, 1909.

[5] Compt. Rend., T. 146, 1908, p. 1024, and T. 147, p. 65, 1908.

[6] Compt. Rend., T. 147, 1908, p. 1024.

[7] Le Radium, T. 6, p. 203, July, 1909.

(2) The actual displacement in a given time of a particle acted on by a constant force.

(3) The actual time of fall through a fixed distance of a particle acted on by a constant force.

In the second part experimental results are given which verify the equations of Part I., and which make it possible to determine directly the value of Ne for gaseous ionization.

PART I. THEORETICAL CONSIDERATIONS.

§ 1. *Distribution of the displacements in successive equal intervals of time of a small spherical particle which is under the influence of no outside forces except the impacts of the gas molecules.*

Since the momentum imparted to the particle by the gas molecules is different in different directions, and changes with the time, the displacement in a given time will continually change its direction and magnitude. To find the chance that a displacement between x and $x+dx$ will occur in a time t, let us use an analysis similar to that which was first used by Maxwell to determine the distribution of molecular velocities.

Let $f(x, y, z)dxdydz$ be a function which gives the chance that a displacement lies between (x, y, z) and $(x+dx, y+dy, z+dz)$. As this chance must be the same in all directions, $f(x, y, z)$ must be invariant under a transformation of axes. By a well-known theorem of invariants, we know that the only function satisfying this condition is $f(x^2+y^2+z^2)$, *i. e.*, a function of the radius vector.

If $F(x)dx$ is the chance that the x component of the displacement is between x and $x+dx$, then the chance that the y component is between y and $y+dy$, and that the z component is between z and $z+dz$, is $F(y)dy$ and $F(z)dz$, respectively. Then, assuming[1] that the x component of the displacement is independent of y and z, we have

$$f(x, y, z)dxdydz = f(x^2+y^2+z^2)dxdydz = F(x)dx \cdot F(y)dy \cdot F(z)dz.$$

Solving,

$$f(x^2) = A^2F(x),\ f(y^2) = A^2F(y),\ f(z^2) = A^2F(z),$$

where

$$A = F(0).$$

Then

$$f(x^2+y^2+z^2) = \frac{1}{A^6} f(x^2) \cdot f(y^2) \cdot f(z^2).$$

Substitute

$$\log \left\{ \frac{1}{A^3} f(x^2) \right\} = H(x^2).$$

[1] Boltzman objected to a similar assumption in Maxwell's work, but obtained the same formula by a more rigorous analysis.

Then $$H(x^2+y^2)=H(x^2)+H(y^2).$$

Therefore

$$H(x^2)=-\alpha x^2$$

and

$$H(y^2)=-\alpha y^2$$

and our required solution is

$$f(x, y, z)=A^3e^{-\alpha(x^2+y^2+z^2)}. \tag{1}$$

Also

$$F(x)dx=Ae^{-\alpha x^2}dx, \quad F(y)dyAe^{-\alpha y^2}dy, \quad F(z)dz=Ae^{-\alpha z^2}dz. \tag{2}$$

As (2) gives the probability that a displacement lies between x and $x+dx$, then, if η displacements of a single particle are observed, the number which will probably lie between x_1 and x_2 is given by

$$n=\eta\int_{x_1}^{x_2}Ae^{-\alpha x^2}dx. \tag{3}$$

A and α are functions which are independent of x, and are determined by the equations:

$$\eta=\eta A\int_{-\infty}^{\infty}e^{-\alpha x^2}dx=A\eta\sqrt{\frac{\pi}{\alpha}}$$

and

$$(x^2)_a{}^*=A\int_{-\infty}^{\infty}x^2e^{-\alpha x^2}dx=A\tfrac{1}{2}\alpha^{-\frac{3}{2}}\sqrt{\pi},$$

which give

$$A=\sqrt{\frac{\alpha}{\pi}}, \tag{4}$$

$$\alpha=\frac{1}{2(x^2)_a}. \tag{5}$$

It is plain that we can interpret η in the above formula as referring to η similar particles having displacements in one and the same interval of time t, and the same analysis will hold. With such an interpretation $(x^2)_a$ is a group average, that is to say, all the particles having displacements between o and Δx are taken in one group: those between Δx and $2\Delta x$ in a second group, and so on. It seems reasonable to suppose that the average square of the displacement during a time t of η similar particles obtained in this way is the same as the average square of the displacement of one single particle during η successive intervals of time t, and this supposition will be made in what follows.

The average displacement along any axis is zero, but the average

*In this work the subscript a has been used to denote an average value, and the subscript p to denote the most probable value.

absolute value of the displacement, that is, the average of the positive or negative displacements is

$$x_a = 2\sqrt{\frac{\alpha}{\pi}}\int_0^\infty xe^{-\alpha x^2}dx = \sqrt{\frac{2}{\pi}}\sqrt{(x^2)_a}. \tag{6}$$

In order to find the absolute displacement in space, change $f(x, y, z)$ in (1) to spherical coördinates by the transformations

$$r^2=x^2+y^2+z^2; \quad x=r\cos\theta; \quad y=r\sin\theta\cos\varphi; \quad z=r\sin\theta\sin\varphi.$$

Then the number out of η displacements that lie between r and $r+dr$ is

$$dn = \eta\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} r^2e^{-\alpha r^2}dr\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta d\theta,$$

or

$$dn = \eta 4\pi\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} r^2e^{-\alpha r^2}dr. \tag{7}$$

This function has a maximum at $\alpha r^2=1$. Therefore the most probable displacement is given by

$$r_p = \sqrt{2}\sqrt{(x^2)_a}. \tag{8}$$

The average displacement in space is

$$r_a = 4\pi\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}}\int_0^\infty rr^2e^{-\alpha r^2}dr = 2\sqrt{\frac{2}{\pi}}\sqrt{(x^2)_a}. \tag{9}$$

which shows that $r_a=2x_a$.

It is also useful for comparison with observation to find an expression for the average displacement in a plane. The chance that the projection of the displacement on the xy plane is between (x, y) and $(x+dx, y+dy)$ is $F(x)dx\cdot F(y)dy$, or

$$\frac{\alpha}{\pi}e^{-\alpha(x^2+y^2)}dxdy.$$

Now transform by the relations

$$x=R\cos\theta \text{ and } y=R\sin\theta,$$

and we get for the number out of η displacements that lie between R and $R+dR$

$$dn = \eta\frac{\alpha}{\pi}Re^{-\alpha R^2}dR\int_0^{2\pi}d\theta,$$

or

$$dn = 2\eta\alpha Re^{-\alpha R^2}dR. \tag{10}$$

This shows a maximum at $\alpha R^2=\frac{1}{2}$, or the most probable value of the displacements in a plane is

$$R_p = \sqrt{(x^2)_a}. \tag{11}$$

The average projection of the displacement on a plane is

$$R_a = 2\alpha \int_0^\infty R^2 e^{-\alpha R^2} dR = \sqrt{\frac{\pi}{2}(x^2)_a}. \tag{12}$$

So we see that if the value of $(x^2)_a$ can be determined, the distribution of the displacements, the average and most probable values of the displacements along an axis, in a plane, or in space can be calculated.

§ 2. *Resistance offered by a gaseous medium to a spherical particle moving uniformly through it.*

Before obtaining an expression for $(x^2)_a$ from dynamical considerations, it is necessary to consider the law of motion of a small particle moving through the gas. Stokes gave an approximate solution to this problem by starting from the fundamental equations of hydrodynamics. However, in solving, it was necessary to make the following assumptions and approximations.

1. A continuous medium.
2. An incompressible fluid.
3. Uniform density.
4. No impressed force upon the gas.
5. The mathematical approximation that the square of the velocity of the gas at any point is negligible.
6. No slip at the surface of the particle.

If F is the force acting upon a particle of radius a, moving with a constant velocity v, through a gas whose coefficient of viscosity is μ, then, with the above assumptions,

$$F = 6\pi\mu a v. \tag{13}$$

If we assume that there is external friction at the surface of the particle, the hydrodynamical[1] equations show that

$$F = 6\pi\mu a k' v, \tag{14}$$

where

$$k' = \frac{1 + \frac{2\mu}{\beta a}}{1 + \frac{3\mu}{\beta a}}, \tag{15}$$

β being the coefficient of external friction. Or, if the coefficient of slip, defined by the equation

$$\rho = \frac{\mu}{\beta}$$

[1] Bassett's Hydrodynamics.

be introduced,

$$k' = \frac{1 + 2\frac{\rho}{a}}{1 + 3\frac{\rho}{a}},$$

and if we can neglect $(\rho/a)^2$ in comparison to ρ/a,

$$k' = \frac{1}{1 + \frac{\rho}{a}}. \tag{16}$$

In order to take account of the molecular structure of the gas, Cunningham[1] has modified Stokes's formula by considering the impacts of the gas molecules upon the particle and has deduced from the theory of probability that

$$F = 6\pi\mu akv, \tag{17}$$

where[2]

$$k = \left(1 + \frac{\frac{3}{2}\frac{l}{a}}{f - 2(f-1)}\right)^{-1}, \tag{18}$$

l being the mean free path of the gas molecule, and f the fraction of the total number of impacts which are elastic. Experiments[3] on oil drops suspended in air at atmospheric pressure show that, at least for the range investigated, if f is put equal to zero, this formula agrees fairly well with the experimental results. Then (18) reduces to

$$k = \frac{1}{1 + \frac{3}{4}\frac{l}{a}}. \tag{19}$$

It is seen that these two formulæ become the same if we place

$$\rho = \tfrac{3}{4}l.$$

From considerations of the kinetic theory, Meyer[4] has deduced the formula for the coefficient of slip

$$\rho = \frac{4}{B}0.3097l,$$

where B is a number between 0 and 2 depending upon the loss of velocity

[1] Proc. Roy. Soc., Vol. A 83, No. A 563, 1910, page 357.
[2] See note 3, page 380, PHYS. REV., April, 1911.
[3] Millikan, PHYS. REV., April, 1911.
[4] Meyer, Die Kinetische Theorie der Gase, p. 211.

of the molecule upon hitting the surface of the particle. Maxwell[1] gives for the inelastic case

$$\rho = \tfrac{2}{3}l.$$

So we see that either Basset's hydrodynamical formula or Cunningham's probability formula is sufficient to represent the experimental facts as long as l/a is small compared to 1. But when l/a becomes greater than 1, they are very different, for in the former k always lies between 1 and $\frac{2}{3}$, while in the latter, k lies between 1 and 0.

In either case, if m is the mass of the particle, then its velocity of steady fall is

$$v = \frac{mg}{6\pi\mu ak}.$$

§ 3. *The average square of the displacement due to molecular impacts.*

If X is the x component of the forces due to the unbalanced molecular impacts of the gas molecules, then the equation of motion along the x axis is

$$m\frac{d^2x}{dt^2} = -F + X,$$

or

$$m\frac{d^2x}{dt^2} = -6\pi\mu ak\frac{dx}{dt} + X. \tag{21}$$

Multiply by x and substitute $d(x^2)/dt = z$, and $dx/dt = v$ and we get

$$\frac{m}{2}\frac{dz}{dt} + 3\pi\mu akz = mv^2 + Xx.$$

If η similar equations are written representing the motion at the same instant of time of η similar particles, then grouped and averaged as indicated in § 1 there results

$$\frac{m}{2}\frac{dz_a}{dt} + 3\pi\mu akz_a = \frac{m}{\eta}\sum_1^\eta v_n^2 + \frac{1}{\eta}\sum_1^\eta X_n x_n.$$

It is obvious that when η is very large

$$\Sigma Xx = 0,$$

as then there would be a group of particles having a negative $X_n x_n$ for every one having a positive $X_n x_n$. Also, since the average kinetic energy of a single particle taken over a long period of time may be taken to be the same as the average kinetic energy of a great many similar particles at any instant of time, $\frac{1}{2}\frac{m}{\eta}\sum_1^\eta v_n^2$ is equal to the average energy of

[1] Maxwell's Scientific Papers, Vol. II., p. 709.

translation in the direction of x of any one of the particles during a long interval of time. If ϵ is the average energy of agitation of the gas molecule and we assume that, on the average, this energy is distributed equally among three rectangular components; then, according to the law of equal partition of energy,

$$\frac{1}{2}\frac{m}{\eta}\sum_{1}^{n} v_n^2 = \frac{1}{3}\epsilon. \tag{22}$$

With these assumptions (21) reduces to

$$\frac{m}{2}\frac{dz_a}{dt} + 3\pi\mu a k z_a = \frac{2}{3}\epsilon,$$

the solution of which is

$$z_a = \frac{2}{3}\epsilon\frac{1}{3\pi\mu a k} + Ae^{-\frac{6\pi\mu a k t}{m}}.$$

The last term is negligible for any time large enough to measure. Therefore,

$$(x^2)_a = \frac{2\epsilon t}{9\pi\mu a k}, \tag{23}[1]$$

where the displacement is measured from the origin.

It is plain from the manner in which the formula for $(x^2)_a$ was derived that it represents the average square of the displacement along the x axis during a time t of a larger number of similar particles, which is taken as the same as the average square of the displacement of a single particle during a larger number of successive equal intervals of time t.

Since $\sum Xx$ necessarily oscillates about zero as n increases it will not only be negligible for n infinite but for many finite values of n and for each of these a solution similar to (23) would be obtained, but the coefficient of t, in general, would be different. Physically, this only means that the average of one series of displacements would not be exactly the same as that of another unless the two series contain an infinite number of displacements.

If the value of $(x^2)_a$ is placed in the formula deduced in § 1, we have the following six forms giving the magnitude of the Brownian displacement.

$$\sqrt{(x^2)_a} = \sqrt{2K}, \qquad (23') \qquad\qquad r_a = \frac{4}{\sqrt{\pi}}K, \qquad (9')$$

[1] This formula was first deduced by Einstein (Ann. d. Phys., 17, p. 559, 1905) and given in essentially the same form as above by Langevin (Compt. Rend., 146, 1908, p. 530). The factor k was not in their formulæ.

(6′) $x_a = \frac{2}{\sqrt{\pi}} K,$ (11′) $R_p = \sqrt{2}K,$

(8′) $r_p = 2K,$ (12′) $R_a = \sqrt{\pi}K,$

where

$$K^2 = \frac{\epsilon t}{9\pi\mu ak}.$$

§ 4. *Effect of Brownian movements upon a small spherical particle moving under the influence of a constant outside force.*

Let the actual displacement be divided into two parts, that due to gravity, and that due to the Brownian movements. The former is simply Vt, where V is a constant, and t is the displacement in one second or the velocity due to gravity. The latter on the average is proportional to the square root of t [see (23)]. In order to treat the problem mathematically it will be assumed that each single displacement due to the Brownian movements may be represented by (24)

(24) $$x = u\sqrt{t},$$

u, the displacement in one second due to the Brownian movements, being independent of the time, and varying in different observed values of the displacement in such a way as to give x a distribution represented by (3). This assumption, which amounts to replacing the perfectly disorganized system by a more regular system which has the same distribution and average value of the displacement, seems reasonable, and the results to which it leads may be checked by experiment.

If the value of x from (24) and the value of α from (5) and (23) is substituted in (3), we obtain an expression for the distribution of the values of u, namely

(25) $$u = \eta\sqrt{\frac{h}{\pi}}\int_{u_1}^{u_2} e^{-hu^2}du,$$

where

(26) $$h = \frac{9\pi\mu ak}{4\epsilon}.$$

The average value of u is given by

$$u_a = 2\sqrt{\frac{h}{\pi}}\int_1^\infty ue^{-hu^2}du = \sqrt{\frac{1}{\pi h}},$$

or

(27) $$u_a = 3\sqrt{\pi}\sqrt{\frac{\epsilon}{\pi\mu ak}},$$

which is the same as (6) for $t = 1$, as it should be. Then, if the total displacement is designated by b, we have

$$b = Vt + u\sqrt{t}. \tag{28}$$

To obtain an expression for the distribution of the total displacements in a fixed time, we treat V and t in (28) as constants and by means of this equation change the variable in (25) from u to b. We get for the number out of η displacements that lie between b_1 and b_2

$$n = \eta\sqrt{\frac{h}{\pi t}}\int_{b_1}^{b_2} e^{-\frac{h}{t}(b-Vt)^2}db, \tag{29}$$

$$b_a = \sqrt{\frac{h}{\pi t}}\int_{-\infty}^{\infty} be^{-\frac{h}{t}(b-Vt)^2}db = Vt. \tag{30}$$

This last result could have been obtained directly from (28), since t is constant and the average value of u is zero.

Similarly, to obtain an expression for the distribution of the times of fall through a fixed distance, we treat V and b as constants and change the variable from u to t and obtain for the number of times of fall which lie between t_1 and t_2

$$n = -\eta\sqrt{\frac{h}{\pi}}\int_{t_1}^{t_2}\frac{bt^{-\frac{3}{2}} + Vt^{-\frac{1}{2}}}{2}e^{-\frac{h}{t}(b-Vt)^2}dt,$$

or, putting $t_g = b/V$ and reversing the limits,

$$n = \frac{\eta V}{2}\sqrt{\frac{h}{\pi}}\int_{t_2}^{t_1}(t_g t^{-\frac{3}{2}} + t^{-\frac{1}{2}})\,e^{-\frac{h}{t}(b-Vt)^2}dt. \tag{31}$$

Solving (28) for $\sqrt{t}$ we get

$$\sqrt{t} = \frac{-u + \sqrt{u^2 + 4bV}}{2V},$$

which shows that the limits corresponding to
$u_1 = -\infty$ and $u_2 = \infty$ are $t_1 = \infty$ and $t_2 = 0$ respectively.
Therefore,

$$1 = \tfrac{1}{2}V\sqrt{\frac{h}{\pi}}\int_0^{\infty}(t_g t^{-\frac{3}{2}} + t^{-\frac{1}{2}})\,e^{-\frac{h}{t}(b-Vt)}dt,$$

and

$$t_a = \tfrac{1}{2}V\sqrt{\frac{h}{\pi}}\int_0^{\infty} t(t_g t^{-\frac{3}{2}} + t^{-\frac{1}{2}})\,e^{-\frac{h}{t}(b+Vt)}\,dt = \sqrt{\frac{h}{\pi}}\int_{-\infty}^{\infty} te^{-hu^2}du,$$

or, substituting the value of t in terms of u,

$$t_a = \sqrt{\frac{h}{\pi}} \int_{-\infty}^{\infty} \frac{2u^2 + 4bV - 2u\sqrt{u^2 + 4bV}}{4V^2} \cdot e^{-hu^2} du,$$

which gives

$$t_a = \frac{1}{4hV^2} + \frac{b}{V}, \tag{32}$$

or, solving for V,

$$V = \frac{b}{t_a}\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{t_a}{hb^1}}\right) = V_a\left(1 + \frac{t_a}{4hb^2}\right), \tag{33}$$

$$t_g = t_a\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{t_a}{hb^2}}\right)^{-1} = t_a\left(1 - \frac{t_a}{4hb^2}\right). \tag{34}$$

These formulæ show that, although the average of a great many observed distances of fall during a fixed time is the actual displacement due to gravity, the average of a great many times of fall through a fixed distance is somewhat different from the time of fall due to gravity. The difference, if small, is inversely proportional to the square of the fixed distance.

The value of the integral in (31) may be obtained by graphical means, or preferably from an error function table, since

$$\eta\sqrt{\frac{h}{\pi}} \int_{t_2}^{t_1} \frac{V}{2}\left(t_g t^{-\frac{3}{2}} + t^{-\frac{1}{2}}\right) e^{-\frac{h}{t}(b - Vt)^2} dt = \eta \frac{1}{\sqrt{\pi}} \int_{s_1}^{s_2} e^{-s^2} ds,$$

where

$$s = \sqrt{h}\frac{b - Vt}{\sqrt{t}}.$$

All of the equations in this section can be applied to a particle under the influence of constant forces other than gravity, as for example, a charged particle in a uniform electric field. In this paper V', t', t_a', t_b', etc., will be used in place of the letters given above when they refer to a particle charged with a single elementary charge under the combined influence of gravity and a uniform electric field. Similarly, V'', t'', and V''', t''', etc., will refer to a particle charged with 2, 3, etc., elementary charges.

§ 5. *Effect of Brownian movements upon the apparent value of the electric charge carried by a particle suspended in a gas.*

The method of measuring e which was originated by Professor Millikan[1] consists in getting an isolated charged particle between two condenser plates and observing its velocity V_1 under the influence of gravity, and

[1] Phil. Mag., February, 1909; Science, p. 436, September 30, 1910; Phys. Zeit., No. 24, December, 1910.

then its velocity V_1' when the electric field is on. If E is the strength of the electric field in absolute electrostatic units acting upon the particle of apparent mass m' and density σ; g, the gravitational constant; and ρ, the density of the gas, then the following three equations enable one to express the value of e, the electric charge carried by the particle in terms of measurable quantities.

$$\frac{Ee + m'g}{m'g} = \frac{V_1'}{V}, \tag{35}$$

$$6\pi\mu a k V_1 = m'g, \tag{20'}$$

$$m' = \tfrac{4}{3}\pi(\sigma - \rho)a^3. \tag{36}$$

Eliminating m' and a and solving for e we get

$$e = 9\sqrt{2\pi}\sqrt{\frac{(\mu k)^3}{2(\sigma - \rho)}} \cdot \frac{V^{\frac{1}{2}}(V^1 - V_1')}{E}. \tag{37}$$

(35) and (20') are derived on the assumption that Ee and $m'g$ are the only forces acting upon the drop, in other words, that the X spoken of in § 3 is zero. This is never strictly true, and therefore the observed velocities are variables, even for a particle which does not change its size, shape or the electric charge which it carries. However, (31) shows that the successive times of fall through a distance greater than 1 cm. of a particle having a radius larger than 100×10^{-6} cm., suspended in air at atmospheric pressure, differ from t_g by less than one tenth of a second. In any case, if V and V' are substituted in place of the observed velocities V_1 and V_1', the value of e should be correct and invariable.

PART II. EXPERIMENTAL APPLICATIONS.

§ 1. *Description of apparatus and method of observation.*

The apparatus which was used in these experiments was designed by Professor Millikan and the writer for the sake of testing the laws of motion of a small drop through a gas at reduced pressures; the results of this study are to be published in another paper. As will appear later, the results herein contained are completely independent of these laws. The same condenser plates which were described by Professor Millikan[1] were placed in a brass vessel which could be made air tight. The arrangement is represented in the vertical cross section in Fig. A. D_1 and D_2 represent the condenser plates. M_1 is an electro-magnet which slides the cover F off of a small hole which is bored through the top plate. M_2 is another electro-magnet which lifts the lead screen L, thus exposing

[1] PHYS. REV., April, 1911.

the air chamber between the condenser plates to the ionizing effect of a few milligrams of radium bromide. A is an ordinary atomizer. B_1, B and B_3 are ebonite bushings through the walls of the brass cylinder. P represents a pump which may be used either for exhaustion or compres-

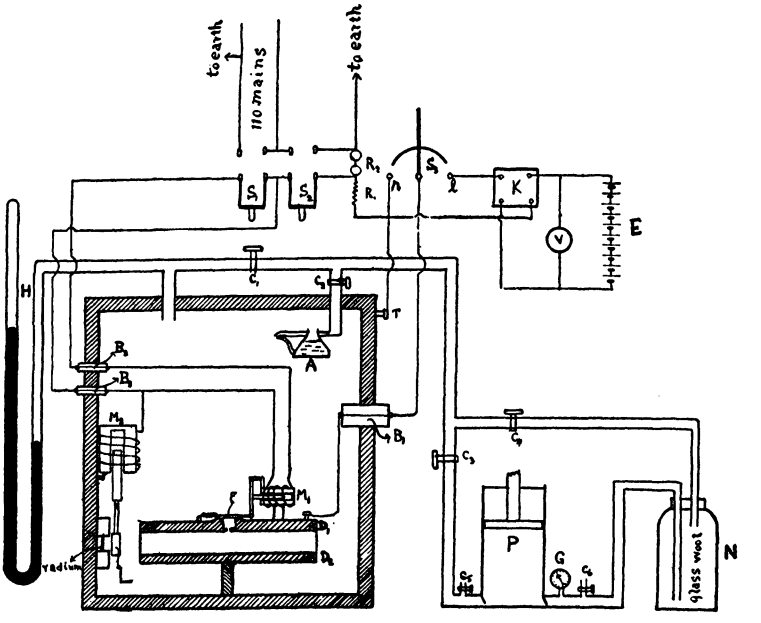


Fig. A.

sion. G is a high pressure gauge and H is a low pressure gauge or simply a U-shaped manometer. All of the air which goes into the brass vessel is made to pass through a vessel N which contains glass wool moistened with sulphuric acid. Also the interior of the large vessel is thoroughly oiled, so that the air with which the atomizer is blown and also the air contained in the vessel is dry and dust-free. C_1, C_2, C_3, C_4, C_5 and C_6 are ordinary stop cocks. The enclosing brass cylinder is connected by means of the binding post T through the resistances R_1, 20 ohms, and R_2, 450 ohms, to earth. The switch S_3, when in contact with the mercury cup r, short circuits the two plates D_1 and D_2 and connects them to earth, and when in contact with the mercury cup l, produces an electrostatic field between the plates, the intensity of which is measured by the voltmeter V. K represents a commutator. The knife edge double pole switches S_1 and S_2 operate the electro-magnets M_1 and M_2 respectively.

The horizontal cross section in Fig. B shows the method which was used to illuminate the small oil droplet. Light from an electric arc A

is made very slightly convergent by a plano-convex lens L, and then passes through a column of water 40 inches long which absorbs practically all of the radiant heat. To prevent the light from falling upon either plate it is passed through a slit s before entering the large cylinder, and then through the plane glass windows W_1 and W_4, falling upon the oil drop near the center of the plate. The illuminated oil drop is observed through the telescope T which is placed at an angle of about 172° with

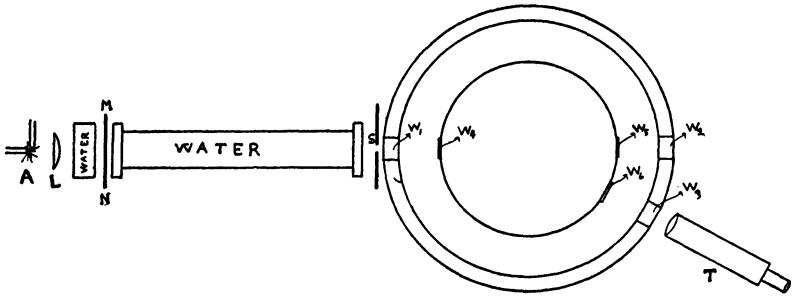


Fig. B.

the incident beam. A scale with seven large divisions subdivided into tenths is placed at the focus of the objective and viewed by the eye-piece. The screen MN serves to cut off all of the light and heat from the arc except when an observation is being taken.

To take a set of observations, C_1, C_2 and C_6 (Fig. A) are opened, and the air is exhausted from the large vessel until the manometer H shows the desired pressure. Then these stop-cocks are closed and C_4 and C_5 are opened and the air is compressed in the vessel N. The cover F is removed by closing the switch S_1; C_2 is opened, and the atomizer produces a fine oily spray which falls through the small hole in the top plate, and which finally comes between the condenser plates into the beam of light, where it can be seen by the aid of the telescope. In this spray of oil the individual drops differ very much in size and the magnitude of the charge which they carry—the smaller showing strong Brownian movements while the larger ones are almost wholly unaffected. Since they would then have different velocities both in the gravitational field and electrostatic field it is possible to select any desired droplet and keep it in the field of view; and, at the same time, rid the field of all the others.

If at any time the droplet should become uncharged and continue to fall after the electrostatic field is turned on, the lead screen L is lifted by closing the switch S_2, thus permitting the radium to produce ions in the space between D_1 and D_2, some of which are soon caught by the droplet. By adjusting the strength of the field, the droplet may be

balanced, or made to move up or down with any speed that is desired. In this way small drops which have shown strong Brownian movements have been held under observation for more than five hours, in which time as high as 411 times of fall or displacements have been taken upon the same drop.

In order to calculate what the distribution of the times of fall or displacements should be, it is necessary to know the value of h. If m' is eliminated from (35) and (20′) we have

$$ak = \frac{Ee}{(V + V')6\pi\mu}, \tag{38}$$

or substituting this value in (26) and remembering that

$$\epsilon = \frac{3}{2}\frac{RT}{N},$$

there results

$$h = \frac{1}{4}\frac{Ne}{RT}\frac{E}{(V + V')}, \tag{39}$$

and

$$u_a = 2\sqrt{\frac{RT}{\pi Ne}}\sqrt{\frac{V + V'}{E}}. \tag{40}$$

This shows that *all the distributions spoken of in Part I., and also the average value of the Brownian displacement in one second can be calculated without knowing the density of the drop, the viscosity or pressure of the gas, or the law of motion of a sphere through a viscous medium. The only assumption involved is that the velocity of steady motion is proportional to the moving force*, an assumption which is fully justified experimentally both here and elsewhere. To obtain u_a and h it is only necessary to know the electrostatic field strength, the velocity of the drop in this field when it carries a single elementary charge, and its velocity when the condenser plates are short-circuited. It is very easy to tell what velocity in the electric field corresponds to a single elementary charge, as has been explained elsewhere.[1] Since Ne/RT is known very accurately, the error in the calculated values of h and u_a depends only upon the accuracy with which E and $V + V'$ can be measured. Therefore at the beginning of each set of observations 10 or 20 times of fall and times of rise through a long distance are taken, which determine V and V' very accurately. Even though V is so small as to be materially affected by the Brownian movements, V' can always be made so large that the error in the average of 10 or 20 values of $V + V'$ is small.

[1] Millikan, PHYS. REV., April, 1911. Millikan and Fletcher, Phil. Mag., July, 1911.

§ 2. *Comparison of the observed and calculated values of the Brownian displacement in a fixed time.*

To observe directly the component of the Brownian displacement along a line, the value of Xe was made equal and opposite to mg so that on the average the drop travelled neither up nor down. The vertical distance of the drop from some reference line in the eyepiece was noted every 10 seconds. Then the difference in two consecutive readings gives

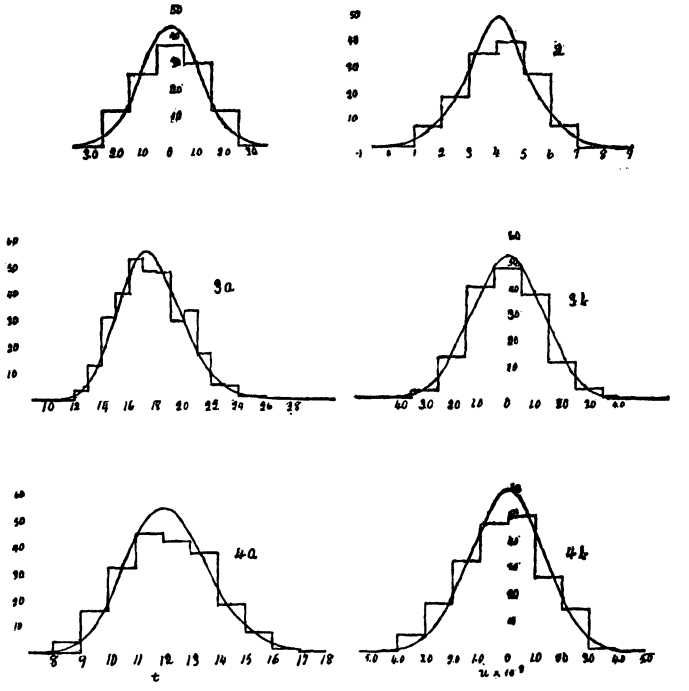


Fig. 1.

directly the Brownian displacement. Table I. shows the results obtained for a single oil drop. In the first part are the data which determine V and V'; in the second part the displacements are arranged not in the order in which they were observed but according to their magnitude. In the last part "obs n" is the number of these displacements which lie between x_1 and x_2 and "cal n" is the value of n obtained from (3). The comparison between "obs n" and "cal n" is shown graphically in curve 1 (Fig. 1). The smooth curve is obtained by plotting values of $\eta \sqrt{\frac{\pi}{\alpha}} e^{-\alpha x^2}$ as ordinates and x as abscissas, while the broken curve is obtained from values of "obs n" in Part III. of Table I. The abscissas are expressed n terms of the eye-piece scale divisions.

TABLE I.

Drop No. 1. *Temperature* = 23° C. *Pressure* = 2.12 *cm.*

t for .522 cm.		t' for .522 cm.		
40.6	39.4	14.1	13.9	V = .01311
40.5	38.6	13.9	14.2	V' = .03725
40.3	41.0	13.6	13.9	Volts = 334.5
39.0	39.0	14.0	14.0	h = 4.05×10^4
39.2	40.7	14.4	14.2	

$x \times .00746$ cm. for 10-second intervals.

+2.39	+1.50	+ .90	+.67	+.15	−.08	−.30	−.75	−1.12	−1.80
2.09	1.50	.90	.67	.15	.15	.30	.75	1.20	1.80
2.02	1.42	.10	.60	.08	.15	.37	.75	1.20	1.87
1.87	1.35	.90	.52	.08	.15	.37	.75	1.20	1.87
1.80	1.27	.90	.52	.08	.22	.37	.82	1.22	1.87
1.74	1.27	.82	.52	.00	.22	.37	.82	1.27	1.94
1.72	1.27	.82	.52	.00	.22	.45	.82	1.27	2.09
1.65	1.20	.75	.52	.00	.22	.45	.82	1.35	2.32
1.65	1.20	.75	.37	.00	.30	.52	.97	1.42	2.46
1.65	1.20	.75	.37	.00	.30	.60	.97	1.57	
1.65	1.05	.75	.37	.00	.30	.60	.97	1.57	
1.50	.97	.75	.30	.00	.30	.67	.97	1.67	
1.50	.90	.75	.15		.30	.67	.97	1.72	

obs $x_a = 8.95 \times 10^{-3}$ cm. cal $x_a = 8.76 \times 10^{-3}$.

$x_1 \times .00746$	$x_2 \times .00746$	cal n	obs n
+2.5	+1.5	9.6	13
+1.5	+0.5	29.7	31
+0.5	−0.5	43.4	38
−0.5	−1.5	29.6	27
−1.5	−2.5	9.6	13

§ 3. *Comparison of the observed and calculated values of the displacement produced in a fixed time when an outside force is acting upon the droplet.*

In this case the drop was allowed to move slowly against g and its displacement in successive 10-second intervals was taken. The results are tabulated in Table II. in a manner exactly like that used in the previous section, "cal n" being taken from (29) and u_a from (28). The graphical comparison is shown in curve 2, Fig. 1.

§ 4. *Comparison of the observed and calculated values of the time of fall through a fixed distance.*

Most of the data were taken by this method because it is capable of giving higher accuracy than that obtainable with either of the other

TABLE II.

Drop No. 2. Temperature = 21° C. *Pressure* = 2.08 *cm.*

t for .522 cm.		t'' for .522 cm.		
28.8	26.0	17.0	17.1	$V = .01933$
27.1	26.3	19.4	16.1	$V'' = .00303$
26.0	27.0	16.2		$V + V' = .01118$
27.5	27.6	17.5		Volts = 107.5
				$h = 5.91 \times 10^4$

$b + .00746$ for 10-second intervals.

1.4	2.5	3.0	3.4	3.8	4.2	4.5	5.0	5.2	5.7
1.5	2.5	3.1	3.4	3.8	4.2	4.5	5.0	5.2	5.9
1.5	2.5	3.1	3.4	4.0	4.2	4.5	5.0	5.3	5.9
1.7	2.6	3.1	3.5	4.0	4.3	4.6	5.0	5.3	6.1
1.8	2.7	3.1	3.5	4.0	4.3	4.6	5.0	5.3	6.2
1.9	2.8	3.2	3.6	4.0	4.3	4.6	5.0	5.4	6.2
1.9	2.8	3.2	3.6	4.0	4.3	4.6	5.1	5.4	6.2
2.0	2.8	3.2	3.6	4.0	4.4	4.7	5.1	5.4	6.3
2.1	2.9	3.2	3.7	4.1	4.4	4.7	5.2	5.4	6.4
2.1	3.0	3.2	3.7	4.2	4.5	4.7	5.2	5.5	
2.1	3.0	3.2	3.7	4.2	4.5	4.8	5.2	5.5	
2.2	3.0	3.3	3.7	4.2	4.5	4.9	5.2	5.5	
2.3	3.0	3.3	3.8	4.2	4.5	4.9	5.2	5.5	
2.5	3.0	3.3	3.8	4.2	4.5	5.0	5.2	5.5	

obs $u_a = 2.365 \times 10^{-3}$. cal $u_a = 2.32 \times 10^{-3}$.

$b_1 + .00746$	$b_2 + .00746$	cal n	obs n
$-\infty$	0	.00	0.0
0	1	.88	0.0
1	2	5.56	7.5
2	3	20.04	18.5
3	4	38.80	35.0
4	5	39.40	39.5
5	6	22.40	27.5
6	7	6.65	8.0
7	$+\infty$	1.32	0.0

methods. In all of the observations recorded in this section, except those relating to Drop No. 8, the variation in the time of fall was taken under the influence of gravity alone. But since (21) shows that V is proportional to $1/ak$ while u_a is proportional to $1/V\overline{ak}$ the Brownian movements would affect the velocity less at a low pressure than at a high pressure. In order to work, however, at a low pressure where the Brownian movements are large, for Drop No. 8 the velocity was made small by superposing an electric field upon the gravitational field. However there is an objection to this procedure since a very slight change in the strength

of the electrostatic field produces a very perceptible change in the velocity of the drop.

The results are tabulated in a manner similar to that used in sections 2 and 3. For example in Table III., Part I., in the column "t'" are given values of the time in seconds required for the singly charged drop to move .522 cm. under the influence of the electrostatic field; and simi-

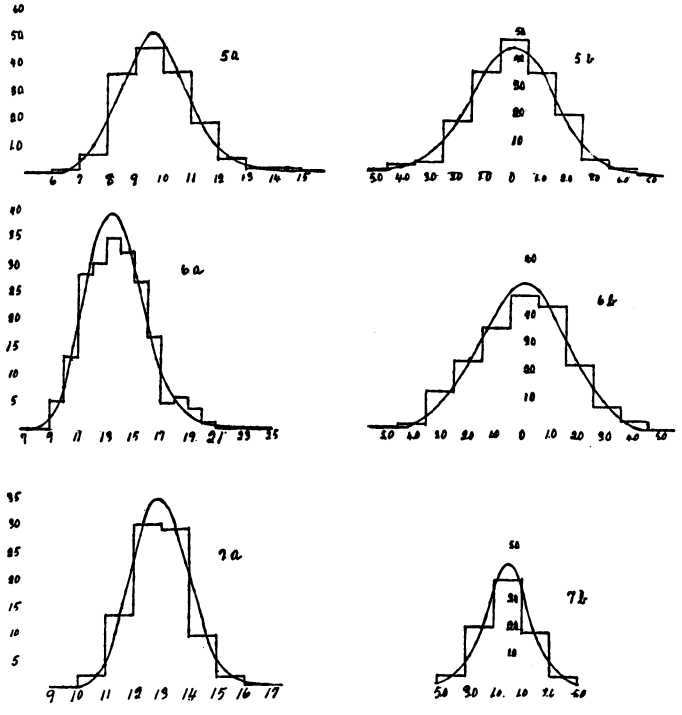


Fig. 2.

larly in the columns, t'', t''' and t^{iv} are given the values of the time for the same drop when it is charged with two, three and four elementary charges. It is seen from the values in the last column that

$$V + V' = \tfrac{1}{2}(V + V'') = \tfrac{1}{3}(V + V''') = \tfrac{1}{4}(V + V^{iv}),$$

as the theory predicts. Therefore the values $\frac{1}{2}(V+V'')$, $\frac{1}{3}(V+V''')$ and $\frac{1}{4}(V+V^{iv})$ were used to determine the mean value of $V+V'$. Also in the last column is given the value of h computed from (39). In part II. are given the successive times of fall through .0373 cm. At the bottom are given the average values obtained from these times of fall. V is computed by (33) and "cal u_a" by (40). The theory predicts that the average component of the displacement in one second along a line

should be $.945\times10^{-3}$ and the displacement observed was $.990\times10^{-3}$. In Part III. is shown the comparison of the observed and calculated

TABLE III.

Drop No. 3. Temperature $=20°$ *C. Pressure* $=23.1$ *cm.*

Part I.			$b=.522$ cm.	Volts $=637$.
t'	t''	t'''	t^{iv}	
58.0	26.0	16.3	11.6	$V=.00213$
59.0	25.5	16.4		$V'=.00881$
58.2	25.7	17.0		$V''=.0205$
59.5	26.3	16.5		$V'''=.0315$
58.4	26.0			$V^{iv}=.0437$
	26.4			Mean $V+V'=.0111$
	25.9			$h=35.6\times10^4$
	26.3			

Part II.			Values of t.				$b=.0373$ cm.		
12.2	14.6	15.6	16.3	17.0	17.5	18.3	19.0	20.0	21.4
12.4	14.6	15.6	16.3	17.0	17.5	18.3	19.0	20.0	21.5
12.5	14.6	15.6	16.3	17.0	17.6	18.3	19.0	20.0	21.4
13.0	14.7	15.7	16.4	17.0	17.6	18.4	19.0	20.1	21.5
13.2	14.8	15.7	16.4	17.0	17.6	18.4	19.0	20.1	21.6
13.4	14.8	15.7	16.4	17.0	17.6	18.4	19.1	20.2	21.7
13.6	14.8	15.7	16.4	17.0	17.7	18.4	19.2	20.2	21.7
13.7	14.8	15.7	16.4	17.0	17.7	18.4	19.2	20.2	21.8
13.8	14.8	15.7	16.4	17.0	17.7	18.4	19.2	20.3	22.0
13.8	14.8	15.8	16.5	17.0	17.7	18.5	19.4	20.3	22.2
13.8	14.9	15.8	16.5	17.0	17.8	18.5	19.4	20.4	22.2
13.9	14.9	15.8	16.5	17.1	17.8	18.5	19.4	20.4	22.5
14.0	14.9	15.8	16.6	17.1	17.8	18.5	19.4	20.4	22.5
14.0	15.0	15.8	16.6	17.1	17.8	18.5	19.4	20.4	22.6
14.0	15.0	16.0	16.6	17.1	17.8	18.6	19.4	20.4	23.2
14.0	15.0	16.0	16.6	17.1	17.8	18.6	19.4	20.4	23.3
14.0	15.0	16.0	16.7	17.2	17.8	18.6	19.5	20.4	23.3
14.0	15.1	16.0	16.7	17.2	17.8	18.6	19.5	20.4	23.3
14.0	15.1	16.0	16.7	17.2	18.0	18.6	19.5	20.4	23.3
14.0	15.2	16.0	16.7	17.2	18.0	18.6	19.5	20.4	24.3
14.0	15.2	16.0	16.7	17.2	18.0	18.6	19.5	20.4	25.4
14.1	15.3	16.1	16.7	17.2	18.0	18.7	19.6	20.6	26.3
14.2	15.3	16.2	16.7	17.3	18.0	18.7	19.6	20.6	20.9
14.2	15.3	16.2	16.7	17.3	18.0	18.7	19.6	20.6	21.0
14.3	15.4	16.2	16.7	17.4	18.0	18.8	19.7	20.6	21.0
14.3	15.4	16.2	16.8	17.4	18.0	18.8	19.8	20.7	21.1
14.4	15.4	16.2	16.8	17.4	18.1	18.8	19.8	20.7	21.3
14.4	15.5	16.2	16.8	17.4	16.1	18.8	19.9	20.7	21.3
14.5	15.5	16.3	16.8	17.4	18.1	18.8	19.9	20.7	
14.5	15.5	16.3	16 8	17.4	18.1	18.8	19.9	20.8	
14.5	15.5	16.3	16.9	17.5	18.2	18.8	20.0	20.8	
14.6	15.6	16.3	16.9	17.5	18.2	18.8	20.0	20.8	

$t_a = 17.65$, $V_a = .00211$, $V = .00213$, $\eta = 321$, obs $u_a = .990\times10^{-3}$, cal $u_a = .945\times10^{-3}$.

Part III. Distribution of t.				Distribution of u.			
t_1	t_2	obs n	cal n	u_1	u_2	obs n	cal n
0	12	0.0	.69	$+\infty$	+3.5	0	.5
12	13	3.5	3.3	+3.5	+2.5	5	5.3
13	14	13.0	10.9	+2.5	+1.5	31	27.1
14	15	31.0	24.2	+1.5	+0.5	74	75.3
15	16	35.0	40.8	+0.5	−0.5	98	105.0
16	17	53.0	52.4	−0.5	−1.5	78	75.3
17	18	48.5	54.7	−1.5	−2.5	27	27.1
18	19	48.0	47.7	−2.5	−3.5	7	5.3
19	20	29.5	35.6	−3.5	$-\infty$	1	.5
20	21	33.5	23.5				
21	22	12.5	13.7				
22	23	5.5	7.3				
23	24	5.0	3.6				
24	25	1.0	1.6				
25	26	1.0	.69				
26	∞	1.0	.47				

TABLE IV.

Drop No. 4. Temperature = 21° *C. Pressure* = 14.3 *cm.*

Part I.			b = .522 cm.			Volts = 483.5.	
t			t'		t''	t'''	
							V = .00307
160	166	182.8	73.5	68.7	29.7	18.4	V' = .00742
158	187	175.2	71.0	70.2	28.5	18.9	V'' = .01754
158.9	178	164.4	72.8	67.4	29.1	18.8	V''' = .0279
167.0	175.6	178.0	71.0		29.0	t^{iv}	V^{iv} = .0387
161.5	176.4	169.2	68.0			13.5	V^{v} = .0607
166.2	168.0		70.0			t^{iv}	Mean $V + V'$ = .01041
170.0	170.0		70.4			8.6	$h = 28 \times 10^4$

t_a = 12.12, V_a = .00308, V = .00310, η = 202, obs $u_a = 1.061 \times 10^{-3}$, cal $u_a = 1.057 \times 10^{-3}$.

Part III. Distribution of t.				Distribution of u.			
t_1	t_2	obs n	cal n	u_1	u_2	obs n	cal n
0	8	0	0	$+\infty$	+4.0	0	.24
8	9	4.0	1.4	+4.0	+3.0	6	2.1
9	10	16.0	10.5	+3.0	+2.0	18	11.8
10	11	32.0	31.4	+2.0	+1.0	34	32.2
11	12	44.5	50.7	+1.0	0.0	48	55.5
12	13	42.0	50.7	0.0	−1.0	51	55.5
13	14	37.5	33.9	−1.0	−2.0	28	32.2
14	15	18.0	16.2	−2.0	−3.0	16	11.8
15	16	7.5	6.3	−3.0	−4.0	1	2.1
16	17	1.0	1.6	−4.0	$-\infty$	0	.24
17	18	.5	.1				

distributions of the values of t and u. Curves 3*a* and 3*b*, which are plotted as explained in section 2, show graphically this same comparison. In all of the following tables the drop no. and the curve no. correspond. Table III. is complete, but in Tables IV.–IX. the observed values of t in Part II. have been omitted.

TABLE V.

Drop No. 5. Temperature = 20° C. Pressure = 9.97 cm.

Part I.			$b = .522$ cm.				Volts = 318.
t			t'	t''	t'''	t^{iv}	
127.2	132.2	135.5	107.8	38.0	22.7	17.8	$V = .003911$
129.7	132.2	136.1	106.1	38.2	23.2	17.0	$V' = .004790$
129.9	132.3	136.4	108.0	38.5	23.2	17.2	$V'' = .01352$
130.2	132.4	137.8	106.5	38.3	24.1	17.1	$V''' = .02222$
130.8	133.2	138.4	109.0	38.4	23.8	t^{v}	$V^{iv} = .03022$
131.4	133.7	138.8	113.0	39.4	23.9		$V^{v} = .03880$
131.4	134.2	139.5	113.0	39.6		13.3	Weighted mean $V + V'$
132.2	134.7		108.8			13.8	$= .0869$
							$h = 22.7 \times 10^4$

$t_a = 9.72$, $V_a = .00383$, $V = .00388$, $\eta = 171$, obs $u_a = 1.162 \times 10^{-3}$, cal $u_a = 1.183 \times 10^{-3}$.

Part III. Distribution of t.				Distribution of u.			
t_1	t_2	obs n	cal n	u_1	u_2	obs n	cal n
0	7	1.	.7	$+\infty$	+3.5	2	1.6
7	8	6.5	10.5	+3.5	+2.5	3	6.3
8	9	35.5	30.7	+2.5	+1.5	18	18.2
9	10	45.0	48.2	+1.5	+0.5	36	37.3
10	11	36.0	36.0	+0.5	−0.5	48	44.2
11	12	17.5	16.0	−0.5	−1.5	36	37.3
12	13	4.5	4.6	−1.5	−2.5	21	18.2
13	14	1.0	1.1	−2.5	−3.5	5	6.3
14	∞	1.0	0.2	−3.5	$-\infty$	2	1.6

TABLE VI.

Drop No. 6. Temperature = 20° C. Pressure = 7.0 cm.

Part I.			$b = .521$ cm.	Volts = 828.	
t	t'		t''		
The value of V	24.5	25.0	11.2	$V = .00277$	
is obtained	24.6	24.7	11.5	$V' = .0210$	$h = 21.7 \times 10^4$
from data in	24.6	24.6	11.7	$V'' = .0446$	
Part II.	24.3	25.6	11.9		
	24.7	25.2	12.0	Weighted mean $V + V' = .02373$	
	24.6	24.6			

$t_a = 13.78$, $V_a = .00274$, $V = .00277$, $\eta = 200$, obs $u_a = 1.256 \times 10^{-3}$, cal $u_a = 1.215 \times 10^{-3}$.

Part III. Distribution of t.				Distribution of u.			
t_1	t_2	obs n	cal n	$u_1 \times 10^3$	$u_2 \times 10^3$	obs n	cal n
0	9	0	0.3	$+\infty$	+3.5	1	2.1
9	10	5	1.8	+3.5	+2.5	13	7.8
10	11	13	6.8	+2.5	+1.5	24	22.3
11	12	28	16.7	+1.5	+0.5	36	42.0
12	13	30	28.1	+0.5	−0.5	48	51.6
13	14	34.5	35.3	−0.5	−1.5	44	42.0
14	15	32.0	25.0	−1.5	−2.5	23	22.3
15	16	26.5	28.9	−2.5	−3.5	8	7.8
16	17	16.5	20.5	−3.5	$-\infty$	3	2.1
17	18	4.5	13.8				
18	19	5.5	7.2				
19	20	3.5	3.6				
20	21	1.5	1.7				
21	22	0.5	0.8				
22	∞	0.0	0.5				

TABLE VII.

Drop No. 7. Temperature = 20° *C. Pressure* = 6.5 *cm.*

Part I.			$b = .521$.		Volts = 668.
t	t'		t''	t	
Value of V is obtained from data in Part II.	33.7	35.6	14.2		$V = .00579$.
	33.0	34.7	14.4	9.1	$V' = .01417$.
	32.8	33.0	14.7	9.3	$V'' = .0355$.
	34.4	35.4			$V''' = .0566$.
	35.8	34.0		$\overline{t}$	$V^{iv} = .0755$.
	34.9				$h = 20.2 \times 10^4$
				$\overline{6.9}$	Weighted mean $V + V' = .02024$.

$t_a = 12.89$, $V_a = .00577$, $V = .00579$, $\eta = 84$, obs $u_a = 1.278 \times 10^{-3}$, cal $u_a = 1.252 \times 10^{-3}$.

Part III. Distribution of t.				Distribution of u.			
t_1	t_2	obs n	cal n	u_1	u_2	obs n	cal n
0	10	0.0	0.04	$+\infty$	+5	0	.06
10	11	2.0	1.6	+5	+3	3	2.3
11	12	13.0	13.2	+3	+1	21	19.7
12	13	29.5	31.2	+1	−1	38	40.0
13	14	28.5	26.5	−1	−3	19	19.7
14	15	9.0	9.5	−3	−5	3	2.3
15	16	2.0	1.7	−5	$-\infty$	0	.06
16	∞	0.0	0.2				

Particular attention is to be called to Drop No. 8 which showed variations in the time of fall from 2 seconds to 25 seconds, the average velocity being 10 per cent. lower than the actual velocity due to the constant outside force. In this rather extreme case the observed and calculated values show a very good agreement.

TABLE VIII.

Drop No. 8. Temperature = 20° C. Pressure = 2.98 cm.

Part I.		$b = .522$ cm.		Volts = 162.7.
t		t'	t'''	
43.5	49.0	32.5	17.5	$V = .01112$
47.5	49.8	32.0	18.0	$V'' = .01592$
43.4	49.0	32.6	18.5	$V''' = .02888$
45.1	45.0	33.2	18.0	Mean $V + V' = .01346$
45.1	47.6	33.2	18.4	$h = 7.52 \times 10^4$
47.4	47.2	33.4		
47.4	48.6	33.4		
47.4	48.4			

$t_a = 7.29$, $V_a' = .00205$, $V' = .00226$, $\eta = 411$, obs $u_a = 1.95 \times 10^{-3}$, cal $u_a = 2.05 \times 10^{-3}$.

Part III. Distribution of t'.				Distribution of u.			
t_1'	t_2'	obs n	cal n	u_1	u_2	obs n	cal n
0	2	.5	0.9	+8	+6	2	3.7
2	3	9.0	13.4	+6	+4	17	20.5
3	4	35.0	38.1	+4	+2	70	65.2
4	5	57.5	57.7	+2	0	108	115.2
5	6	55.5	60.8	0	−2	113	115.2
6	7	60.5	56.1	−2	−4	53	65.2
7	8	50.5	47.5	−4	−6	18	20.5
8	9	50.0	36.8	−6	−8	5	3.7
9	10	26.0	28.4				
10	11	15.5	21.3				
11	12	17.0	14.4				
12	13	12.5	10.7				
13	14	8.5	7.5				
14	15	4.0	5.5				
15	20	6.5	9.4				
20	∞	2.0	2.1				

TABLE IX.

Drop No. 9. Temperature = 20° C. Pressure = 2.28 cm.

Part I.			$b = .521$ cm.		Volts = 353.
t	t'		t''	t	
The value of	16.8	17.8	8.0	5.4	$V = .00805$
V is ob-	16.8	18.4	7.9	5.0	$V' = .03024$
tained	17.0	17.6	8.0	4.8	$V'' = .0657$
from data	16.8	17.0	7.8	4.8	$V''' = .1042$
in Part II.	16.6				$h = 409 \times 10^4$
					Weighted mean $V + V' = .0378$

$t_a = 9.35$, $V_a = .00795$, $V = .00805$, $\eta = 85$, obs $u_a = 2.43 \times 10^{-3}$, cal $u_a = 2.38 \times 10^{-3}$.

Part III. Distribution of t.				Distribution of u.			
t_1	t_2	obs n	cal n	u_1	u_2	obs n	cal n
0	6	0	0	$+\infty$	+8	0	.9
6	7	1	.8	+8	+6	1	2.6
7	8	9	8.4	+6	+4	8	7.0
8	9	27	24.0	+4	+2	9	13.3
9	10	25	27.8	+2	0	27	18.5
10	11	16	16.8	0	−2	19	18.5
11	12	4	5.7	−2	−4	10	13.3
12	13	3	1.4	−4	−6	8	7.0
13	∞	0	0.2	−6	−8	3	2.6
				−8	$-\infty$	0	.9

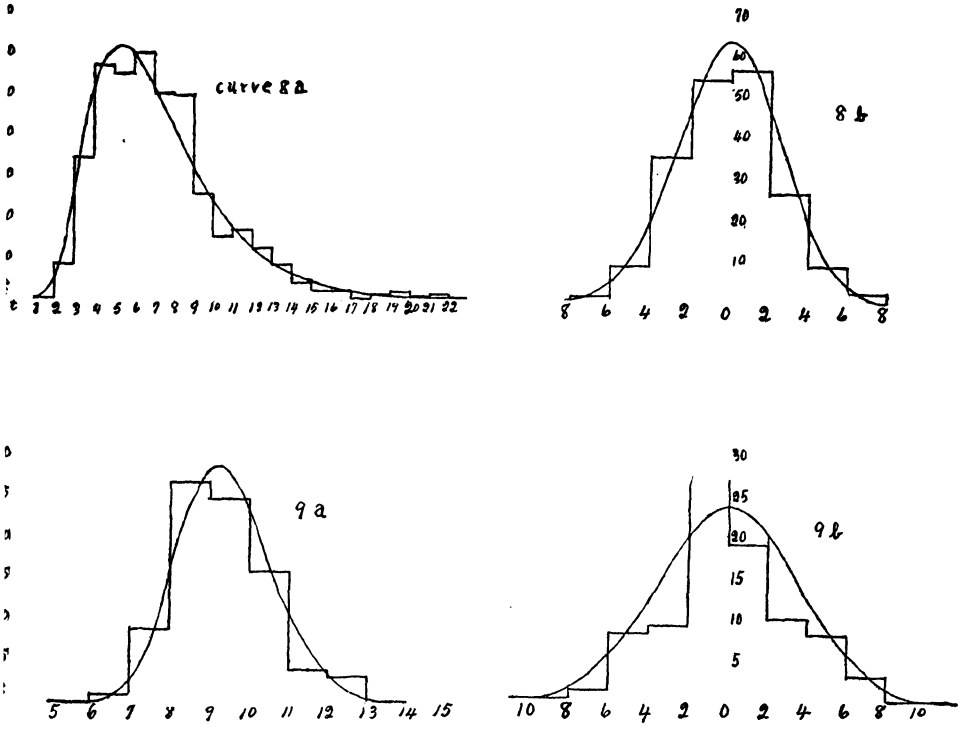


Fig. 3.

Before the condenser plates were put into the large brass cylinder described above, 341 observations were taken upon a single oil drop. 151 values of t were taken for b = .00744 cm.; 155 values for $b = 2\times$ .00744 cm.; and 34 values for $b = 5\times.00744$ cm. The comparison of the observed and calculated distributions for these three cases is shown in curves 10*a*, 10*b*, and 10*c*. The observed values of u_a were $.792\times 10^{-3}$, $.846\times 10^{-3}$ and $.725\times 10^{-3}$. The comparison of the observed and

calculated 341 values of u is shown in Curve 11. The two broken curves represent two different groupings of the same values of u.

Since for oil drops suspended in air at atmospheric pressure the formula

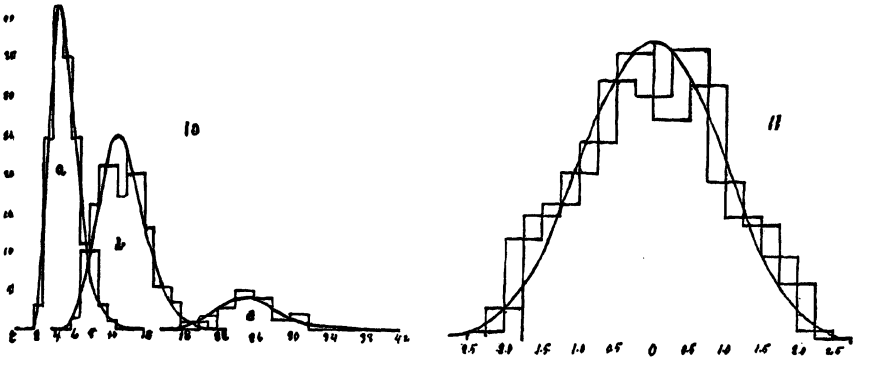


Fig. 4.

$$V = \frac{2}{9}\frac{ga^2(\sigma - e)}{\mu}\left\{1 + .817\frac{l}{a}\right\}$$

has been shown to hold it was used to determine a and k with the result that

$$a = 31.9 \times 10^{-6} \quad \text{and} \quad k = .810.$$

Substituting these values in (27) we get

$$\epsilon = 6.40 \times 10^{-14}$$

and since

$$e = \frac{3}{2}\frac{RT}{N},$$

we get ($G = 296$ and $R = 82.9 \times 10^6$)

$$N = 57.5 \times 10^{22}.$$

We see thus that the Brownian movement theory developed in Part I. accounts completely for the variation in the observed displacements and times of fall, and gives a fairly accurate means of determining molecular magnitudes.

§ 5. *Determination of the value of Ne for gaseous ionization.*

Townsend's and Frank and Westfall's work[1] on mobility and diffusion coefficients is the only evidence, so far, that has justified the assumption that the value of Ne for gaseous ionization is the same as in electrolysis.

[1] For a discussion of their results see Millikan and Fletcher, Phil. Mag., June, 1910.

The fact that the theoretical and observed distributions and the average absolute values of the displacement agree, shows that this assumption is correct. To show how close is this agreement (40) is solved for $\sqrt{Ne}$. The resulting value for each drop is given in Table X.

TABLE X.

Drop No.	$u_a \times 10^3$	T	$V+V'$	Volts.	$\sqrt{Ne} \times 10^{-7}$	η
1	2.83	294	.05036	334.5	1.68	125
2	2.365	294	.01118	107.5	1.67	136
3	.990	293	.1110	637	1.645	321
4	1.061	294	.01041	483.5	1.695	202
5	1.162	293	.00868	318	1.73	171
6	1.256	293	.0237	828	1.65	200
7	1.278	293	.0201	668	1.66	84
8	1.950	293	.133	162.7	1.785	411
9	2.43	293	.0380	353	1.65	85

If weights be assigned according to the number of observations taken as shown in the last column of the above table, there results for *the weighted mean value which represents an average of 1,735 values of u*

$$\sqrt{Ne} = 1.698 \times 10^7$$

or

$$Ne = 2.88 \times 10^{14}.$$

The value of Ne obtained by depositing silver by an electric current, when expressed in electrostatic units is 2.896×10^{14}.

If now the value of N obtained in section 4 is substituted in the above value of Ne we get[1]

$$e = 5.01 \times 10^{-10},$$

which is within 2 per cent. of the value obtained by Millikan. It is to be noticed that *both N and Ne and consequently e are determined by the above method from the same set of experiments and not from totally different types of experiments as heretofore has always been the case, and further that these experiments furnish a direct proof that the value of the elementary charge on the oil drop which it received by catching an ion from the air is the same as the charge on the silver ion in electrolysis.*

§ 6. *Application of the theory to Ehrenhaft's*[2] *and Przibram's*[3] *work on the value of e.*

[1] This shows there is no longer any discrepancy in the values of e obtained by the Millikan method and the Brownian movements method as had appeared to be the case from Perrin's observations on Brownian movements in liquids.

[2] Wien. Ber., 119, 2a, March, 1910.

[3] Wien. Ber., 119, 2a, June, 1910.

In a preliminary report of this work which was published in the Phys. Zeit., VII., 1911, I computed the distribution of the values of t and t' treating the data in Table V. of Ehrenhaft's work as if it were taken all upon one particle. Using Cunningham's formula and assuming the density of the particles to be that of metallic platinum, as Ehrenhaft supposed it to be h was calculated from (26) and found to be given by

$$h = 4.55 \times 10^{4}.^{1}$$

When, however, h is computed from (39), $V+V'$ being computed the values $t_a = 6.86$ and $t_a' = 4.18$, we have

$$h = 23.6 \times 10^{4}.$$

This very great discrepancy can, in my judgment, be explained only by assuming that the density of the particles which he observed was not the density of the metallic platinum. For other reasons Regener[2] has come just recently to the same conclusion.

Using this last value of h we get from (26)

$$ak = 11.44 \times 10^{-6},$$

and from (19)

$$a = 16.56 \times 10^{-6},$$

and from (20)

$$\sigma = 3.3.$$

From this it appears that the density of the platinum particles which Ehrenhaft observed was 3.3 instead of 21.5. The correctness of this statement can be checked if 100 or more observations can be taken upon a single particle. The only data available at present is that taken upon different particles. In curves 12 and 13 (Fig. 5) is shown the comparison between the calculated and observed distribution of t and t'. The smooth curve is the theoretical distribution of the times of fall for a single particle having the average size of those recorded in Ehrenhaft's Table V., while the broken curve gives the observed distribution as taken from this same table. As we would expect there is no such agreement as when the observations are all taken upon the same particle, but the theoretical variation of the values of t and t' is amply sufficient to explain all of Ehrenhaft's irregular values of e. It also indicates that the low value of e is probably due to an error in the density, but this can not be shown conclusively until more observations are taken upon a single particle.

To show the relation of Przibram's work to the present theory a

[1] Due to some minor corrections this value does not agree exactly with the value given in the preliminary report.

[2] Phys. Zeit., XI., 1911.

somewhat different treatment may be used because he obtained two[1] values of t and t' for the same particle. A calculation of the times of fall shows that the variation is not as large as that shown in his tables which only means that the particles were of different sizes. But an idea of the differences in two successive times of fall may be obtained by computing the variation for his average-sized particle.

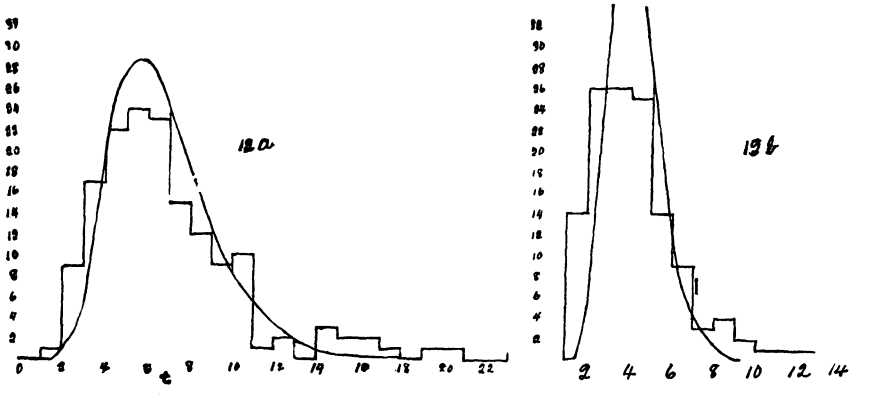


Fig. 5.

For 220 observations taken upon "Phosphornebel" (Table V. of his work) the average time of fall through .0416 cm. is 15.2 seconds. Then

$$V = .00275, \quad a = 44.4 \times 10^6 \text{ cm.}, \quad k = .85, \quad h = 78 \times 10^4,$$

which gives for the probability that a time of fall will be between t_1 and t_2 the equation

$$P = 1.374 \int_{t_1}^{t_2} \frac{t_0 t^{-\frac{3}{2}} + t^{-\frac{1}{2}}}{2} \cdot e^{-\frac{78}{t}(4.15 - 275t)^2} dt.$$

The following values of P were calculated by this formula:

TABLE XI.

t_1	t_2	P
12	13	0.024
13	14	0.13
14	15	0.31
15	16	0.32
16	17	0.17
17	18	0.04
18	19	0.004

[1] Recently both Ehrenhaft and Przibram have taken several observations on a single particle but the number is still too small to make it possible to apply stat stical formulæ.

The probability that any two times occur successively is the product of the probabilities that they will occur separately. For example, the probability that a time between 14 and 15 is followed by a time between 16 and 17 is $.31 \times .17 = .053$. When such a combination occurs the per cent. of difference lies between 14 and 21. Out of 110 double observations 5 or 6 should show a difference of 14 per cent. to 21 per cent. due to the above combinations. Taking all possible combinations the following are obtained:

Between	0	per cent.	and	7	per cent.	about	45	observations.
"	7	"	"	14	"	"	37	"
"	14	"	"	21	"	"	20	"
Above	21	"	"			"	10	"

By calculating the per cent. of difference in two successive times of fall from Przibram's Table V. we have

Between	0	per cent.	and	7	per cent.	55	observations.
"	7	"	"	14	"	"	26	"
"	14	"	"	21	"	"	16	"
Above	21	"	"				13	"

The agreement is sufficiently good to show that by the Brownian movement theory we can account for the differences which he obtained. Also these differences are sufficient to show why he obtained an overlapping of the values of $1e$ and $2e$ or $2e$ and $3e$, etc.

CONCLUSION.

It has been shown that:

1. The continual agitation of a small particle suspended in a gas, known as Brownian movements is entirely explained by the kinetic theory of gases, both qualitatively and quantitatively.

2. The value of Ne obtained from gaseous ionization is the same as that obtained from electrolysis.

3. The irregular values of e apparently resulting from the work of Ehrenhaft and Przibram are entirely explained by the Brownian movement theory.

For this work I am much indebted to Professor R. A. Millikan, who has worked with me while all of the above data were taken, and under whose personal direction this research has been done, and also to Professor A. C. Lunn for his aid and helpful criticism on the first part of this work.

RYERSON PHYSICAL LABORATORY,
April 25, 1911.